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Ignorance, Uncertainty, and Strategic Consumption-Portfolio Decisions*

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Abstract

This paper constructs a recursive utility version of a canonical Merton (1971) model with uninsurable labor income and unknown income growth to study how the interaction between two types of uncertainty due to ignorance affects strategic consumption-portfolio rules and precautionary savings. Specifically, after solving the model explicitly, we theoretically and quantitatively explore (i) how these ignorance-induced uncertainties interact with intertemporal substitution, risk aversion, and the correlation between the equity return and labor income, and (ii) how they jointly affect strategic asset allocation, precautionary savings, and the equilibrium asset returns. Furthermore, we use data to test our model's predictions on the relationship between ignorance and asset allocation and quantitatively show that the interaction between the two types of uncertainty is the key to explain the data. Finally, we find that the welfare costs of ignorance can be very large.

JEL Classification Numbers: *C61, D81, E21.*

Keywords: Ignorance, Unknown Income Growth, Induced Uncertainty, Strategic Asset Allocation.

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1 Introduction

Though most macroeconomic and financial models assume agents have a good understanding of the economic model they use to make optimal decisions, in reality, many ordinary investors/consumers are ignorant about details of the economic model including the structure and parameters of the model and the current state of the model economy their decisions are based on. For example, van Rooija, Lusardic, and Alessie (2010) found that some households do not know the basics of risk diversification when making financial decisions.[1] Mitchell and Lusardi (2014), in a carefully designed survey, found that many respondents, not only in the US but around the world, lack financial literacy, meaning that they do not have necessary skills and knowledge that allows them to make informed and effective investment decisions.[2] Brennen (1998) argued that investors have incomplete information about the investment opportunity set. Guvenen (2007) empirically showed that individuals may not have complete information about their own income growth.[3]

Different types of uncertainty arise due to ignorance about different aspects of the economic model. In a recent paper, Hansen and Sargent (2015) argued that ignorance provides a useful way to summarize different types of uncertainty through specifying the details the decision maker is ignorant about. They examined the implications of ignorance using a simple Friedman (1953) model to "fine tune" an economy. Specifically, they discussed two types of ignorance: (i) the agent is ignorant about the conditional distribution of the next period's state and (ii) the agent is ignorant about the probability distribution of a response coefficient (a parameter) in an otherwise fully trusted specification of the conditional distribution of the next period's state. In other words, the first type of ignorance represents *model uncertainty* (or MU) as the agent does not know the shock distribution, while the second type of ignorance represents *parameter uncertainty* (or PU) as the agent does not know model parameters. Bernanke (2007) also argued that policymakers usually care about the uncertainty about the structure of the economy (including both the transmission mechanism of monetary policy and the model parameters) and the current state of the economy.

Inspired by Hansen and Sargent (2015), in this paper we build a recursive utility version of a basic Merton (1971) model to study the implications of these two major types of ignorance for intertemporal consumption-saving-asset allocation, a fundamental topic in modern macroeconomics and finance. Our central goal is to provide a *unified* framework to study how the

[1] They used the De Nederlandsche Bank (DNB) Household Survey data to study the relationship between financial literacy and stock market participation, and found that financial literacy affects financial decision-making: Those with low literacy are much less likely to invest in stocks.

[2] It is not surprising that people who get any of the three simple questions wrong are unlikely to master more challenging investment strategies.

[3] Although both the expected mean of labor income growth and that of the equity return are crucial for financial decision-making, they are not known *a priori* and are usually estimated with errors.

two major types of ignorance (or the two types of uncertainty induced by ignorance: parameter uncertainty and model uncertainty) interact with each other and then affect the optimal consumption-saving-portfolio decisions as well as the equilibrium asset returns. Specifically, we construct a continuous-time Merton (1971)-Wang (2009) type model with uninsurable labor income and unknown income growth in which the investors have recursive utility and a preference for robustness. In our recursive utility framework, we also disentangle two distinct aspects of preferences: the agent's elasticity of intertemporal substitution (EIS; attitudes towards variation in consumption across time) with the coefficient of absolute risk aversion (CARA; attitudes towards variation in consumption across states), which are shown to have different roles in driving consumption saving and portfolio choice decisions.[4] As explained below, our model deliver not only rich theoretical results but also testable implications.

Hansen and Sargent (1995) first discussed how to model MU due to the preference for robustness (RB) within the linear-quadratic-Gaussian (LQG) economic framework.[5] In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they choose their decisions as if the subjective distribution over shocks was chosen by an evil agent to minimize their utility.[6] In addition, many empirical and experimental studies have repeatedly supported that individual agents are ambiguity averse (or have robustness preferences). For example, Ahn, Choi, Gale, and Kariv (2014) used a rich experiment data set to estimate a portfolio-choice model and found that about 40 percent of subjects display either statistically significant pessimism or ambiguity aversion. Bhandari, Borovička, and Ho (2016) identified ambiguity shocks using survey data, and showed that in the data, the ambiguity shocks are an important source of variation in labor market variables. It is worth noting that canonical consumption-portfolio choice models with uninsurable labor income generally do not distinguish between risk and uncertainty, but emphasize the key role of uncertainty about future labor income in determining strategic consumption-saving-portfolio decisions.

In our model economy, the investors not only have incomplete information about income

[4]Constant-relative-risk-aversion (CRRA) utility functions are more common in macroeconomics, mainly due to balanced-growth requirements. CRRA utility would greatly complicate our analysis because the intertemporal consumption model with CRRA utility and stochastic labor income has no explicit solution and leads to non-linear consumption rules. See Kasa and Lei (2017) for a recent application of RB in a continuous-time Blanchard-Yaari model with CRRA utility and wealth hetergeneity.

[5]See Hansen and Sargent (2007) for a textbook treatment on robustness and model uncertainty.

[6]There are three main ways to model ambiguity and robustness in the literature: the multiple priors model (Gilboa and Schmeidler 1989), the "smooth ambiguity" model (Klibanoff, Marinacci, and Mukerji 2005), and the multiplier utility and robust control/filtering model (Hansen and Sargent 2001). See Epstein and Schneider (2010) for a recent review on this topic. In this paper, we follow the line of Hansen and Sargent to introduce robustness and model uncertainty into our model.

growth but also are concerned about the model misspecification. In other words, they face both parameter uncertainty and model uncertainty. Compared with the full-information case in which income growth is known, parameter uncertainty due to unknown income growth creates an additional demand for robustness. Furthermore, in the standard problem with parameter uncertainty, the agent combines a pre-specified prior over the parameter with the observable variables to construct the perceived value of the parameter, and is assumed to have only a single prior (i.e., no concerns about model misspecification). However, given the difficulty in estimating the mean growth rate of individual income, the sensitivity of optimal decisions to estimation errors, and the substantial empirical evidence that agents are not neutral to ambiguity, it is crucial for us to understand how rational investors facing parameter uncertainty and having multiple priors make robustly strategic decision rules that work well for a set of possible models.

This paper makes four main contributions to the existing literature. First, we provide a unified continuous-time recursive utility framework to explore both the normative and positive implications of parameter and model uncertainty due to ignorance for strategic consumption-portfolio rules in the presence of uninsurable labor income. We show that the optimal consumption/saving-portfolio choice problem under both PU and MU can be solved explicitly. Specifically, it can be formulated by making two additions to the standard full-information rational expectations (FI-RE) model: (i) imposing an additional constraint on the agent's information and knowledge about the mean growth rate of individual income; and (ii) introducing an additional minimization over the set of probability models subject to the additional constraint. The additional constraint recognizes that the probability model of the perceived parameter is not unique. Furthermore, the additional minimization procedure reflects the preference for robustness of the agent who understands that he does not have complete information about the income growth parameter.

Second, after solving the models explicitly, we can inspect the exact mechanism through which these two types of induced uncertainty interact and affect different types of demand for the risky asset and the precautionary saving demand.[7] Specifically, we find that the precautionary saving demand and the strategic asset allocation are mainly affected by the effective coefficient of absolute risk aversion ($\widetilde{\gamma}$) and this coefficient is determined by the interaction between the CARA (γ), the EIS (ψ), and the degree of RB (ϑ) via the formula: $\widetilde{\gamma} = \gamma + \vartheta/\psi$. This expression clearly shows that both risk aversion and intertemporal substitution play roles in determining the amount of precautionary savings and the optimal share invested in the risky asset, but without model uncertainty, only risk aversion matters in determining these two demands. In addition, we show this effective coefficient can affect the parameter learning mechanism. As one testable implication, we show our model can help explain why more educated households hold more risky assets in the

[7] Maenhout (2004), Liu, Pan, and Wang (2005), Liu (2010), Ju and Miao (2012), and Chen, Ju, and Miao (2014) examined how model uncertainty and ambiguity affect portfolio choices and/or asset prices.

data. We quantitatively show that the interaction between the two types of uncertainty is the key to explain the data.

Third, using the explicit consumption-savings-portfolio rules, we show that a general equilibrium under MU and PU can be constructed in the vein of Bewley (1986) and Huggett (1993).[8] We also show that the general equilibrium is unique in such an RU economy with both MU and PU. Furthermore, after calibrating the RB parameter using the detection error probabilities (DEP), we find that RB has significant effects on parameter learning and the interaction of MU and PU has the potential to resolve the risk-free rate puzzle and the equity premium puzzle for plausibly calibrated RB parameter values.

Finally, we show that the welfare cost of ignorance is non-trivial, underscoring the importance of studying parameter uncertainty and model uncertainty arising from different types of ignorance together. Specifically, for a plausibly calibrated robustness parameter, the welfare cost of ignorance could be a significant fraction of total wealth. In addition, we show that the welfare cost is more sensitive to the change in the degree of model uncertainty than to the change in the degree of parameter uncertainty.

This remainder of the paper is organized as follows. Section 2 provides a literature review. Section 3 describes our model setup, introducing key elements step by step. Section 4 presents key theoretical and quantitative results as well as the welfare implications of ignorance. Section 5 provides an extension of the model in which the unknown income growth follows a continuous Gaussian process and discusses the general equilibrium asset pricing implications of ignorance. Section 6 concludes.

2 Literature Review

Our paper is related to two broad branches of literature. First, our paper is related to the broad literature studying consumption-saving and portfolio choices. The recent empirical studies on household portfolios in the U.S. and major European countries have stimulated research in allowing for portfolio choice between risky and risk-free financial assets when households receive labor income and have the precautionary saving motive. The empirical research on household portfolios documents that the stock market participation rate was increasing in the U.S. and Europe and the importance of the precautionary saving motive for portfolio choice. See Guiso, Jappelli, and Terlizzese (1996) and Luigi, Haliassos, and Jappelli (2002). Some recent theoretical studies have also addressed the importance of parameter uncertainty or model uncertainty in

[8] Wang (2003) constructs a general equilibrium under full-information rational expectations (FI-RE) in the same Bewley-Huggett type model economy with CARA utility that we study. Angeletos and Calvet (2006) characterize a closed-form recursive general equilibrium in a neoclassical growth model with idiosyncratic production risk and incomplete markets.

affecting agents' optimal consumption and portfolio rules. For example, Brennen (1998) showed that the uncertainty about the mean return on the risky asset has a significant effect on the portfolio decision of a long-term investor. Maenhout (2004) explored how model uncertainty due to a preference for robustness affects optimal portfolio choice and the equilibrium equity premium. Wang (2009) found that incomplete information about labor income growth can significantly affect optimal consumption-saving and asset allocation.

Second, on the modelling approach, our paper is related to a fast growing literature on modeling uncertainty, both model uncertainty and parameter uncertainty. Besides Hansen and Sargent (2015) as we previously mentioned, this paper is also closely related to Maenhout (2004, 2006), Garlappi, Uppal, and Wang (2007), Wang (2009), and Collin-Dufresne, Johannes, and Lochstoer (2016). Maenhout (2004) explored how model uncertainty due to a preference for robustness reduces the demand for the risky asset and increases the equilibrium equity premium. Maenhout (2006) analyzed the robust portfolio choice problem when the risk premium is mean-reverting and developed a new method to calibrate the robustness parameter. Garlappi, Uppal, and Wang (2007) examined how allowing for the possibility of multiple priors about the estimated expected returns affects optimal asset weights in a static mean-variance portfolio model. Wang (2009) studied the effects of incomplete information about the mean income growth on a consumer's consumption/saving and portfolio choice in an incomplete-market economy. Collin-Dufresne, Johannes, and Lochstoer (2016) studied general equilibrium models with unknown parameters governing long-run growth and rare events, and showed that parameter learning can generate quantitatively significant macroeconomic risks that help explain the existing asset pricing puzzles. Luo (2016) considered state uncertainty (uncertainty about the values of labor income and financial wealth) within an expected utility partial equilibrium model, and found that state uncertainty due to limited capacity does not play an important role in determining strategic asset allocation unless the investors face very tight information-processing constraints.

But our paper is also significantly different from the above papers. Unlike Maenhout (2004), the present paper explores how the interaction of model uncertainty and parameter uncertainty affects the strategic consumption/saving-portfolio decisions in the presence of uninsurable labor income. The model presented in this paper can therefore be used to study the relationship between the correlation between the labor income risk and the equity return risk and the stockholding behavior. Unlike Wang (2009) and Collin-Dufresne, Johannes, and Lochstoer (2016), this paper considers more general concepts of ignorance and induced uncertainty: We not only consider parameter uncertainty but also model uncertainty due to robustness.[9] In addition, un-

[9] Another key difference between our paper and Wang (2009) is that we not only consider the discrete Markovian unknown income growth process as in Wang (2009) but also consider a continuous Gaussian unknown process, and show that these two specifications may lead to distinct interactions between the two types of induced uncertainty;

like Maenhout (2006) and Garlappi, Uppal, and Wang (2007), this paper focuses on incomplete information about the mean income growth, rather than incomplete information about the risk premium, and studies how this type of incomplete information affects the robust consumption and portfolio rules in an intertemporal setting. The key difference between this paper and Luo (2016) is that this paper focuses on examining parameter uncertainty that is more difficult to learn than the state uncertainty discussed in Luo (2016), and examines how it interacts with model uncertainty within a recursive utility framework in which both types of uncertainty interact with intertemporal substitution and risk aversion.

3 The Model Setup

In this section, we lay out our continuous-time consumption-portfolio choice model with recursive utility and two types of ignorance. To help explain the key structure of the model, we will introduce each of the key elements one by one, starting with specifications of the labor income and investment opportunity set, followed by the description of the information set, then the recursive utility, and finally introducing the model uncertainty due to robustness.

To provide an overview, our model is a continuous-time Merton-type model (1971) with uninsurable labor income and unknown income growth. Specifically, we generalize the Wang (2009) model in the following three aspects: (i) rather than using the expected utility specification, we adopt a recursive utility specification; (ii) to better explore the importance of pervasive uncertainty due to ignorance in investors' financial decision-making problem, we not only consider parameter/state uncertainty due to unknown income growth, but also consider model uncertainty due to a preference for robustness; and (iii) we not only consider the Markovian income growth specification but also consider a Gaussian income growth specification. The Gaussian specification can help explore the general equilibrium implications of different types of ignorance. The typical investor in the model economy has recursive utility and makes strategic consumption-saving-asset allocation decisions with pervasive uncertainty due to ignorance. Specifically, we assume that the consumer can access two financial assets: one risk-free asset and one risky asset, and also receive uninsurable labor income.

3.1 Specifications of Labor Income and Investment Opportunity Set

Labor income (y_t) is assumed to follow a continuous-time AR(1) (Ornstein-Uhlenbeck) process:

$$dy_t = (\mu(Z_t) - \rho y_t)\, dt + \sigma_y dB_{y,t}, \tag{1}$$

consequently, they lead to distinct implications for precautionary saving and strategic asset allocation.

where σ_y is the unconditional volatility of the income change over an incremental unit of time, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state,[10] $B_{y,t}$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, and Z_t is a right-continuous Markov chain and is independent of the Brownian motion, $B_{y,t}$. For simplicity, following the literature, we assume that Z_t takes two values, either high (H) or low (L), during a small time interval, Δt, and they represent the good or bad states of the regime-switching macroeconomy, respectively. During Δt, the good state (H) jumps to the bad state (L) with the transition probability $\lambda_1 \Delta t$ and the bad state L jumps to the good state H with the transition probability $\lambda_2 \Delta t$. The transition densities λ_1 and λ_2 determine how persistent each state of the Markov chain is. For convenience, we denote that

$$\lambda(Z_t) = \begin{cases} \lambda_1, & Z_t = H, \\ \lambda_2, & Z_t = L. \end{cases}$$

Consequently, $\mu(Z_t)$ also takes two values, either high (μ_1) when $Z_t = H$ or low $(\mu_2 < \mu_1)$ when $Z_t = L$. If the time-t income growth is high, i.e., $\mu = \mu_1$, the income growth remains high at time $t + \Delta t$, with probability $1 - \lambda_1 \Delta t$ and decreases to μ_2 at $\lambda_1 \Delta t$. Similarly, if the time-t income growth is low, i.e., $\mu = \mu_2$, the income growth remains low at time $t + \Delta t$, with probability $1 - \lambda_2 \Delta t$ and increases to μ_1 at $\lambda_2 \Delta t$. The conditional transition matrix of μ, P, can thus be written as:

$$P = \begin{bmatrix} 1 - \lambda_1 \Delta t & \lambda_1 \Delta t \\ \lambda_2 \Delta t & 1 - \lambda_2 \Delta t \end{bmatrix},$$

which means that the stationary distribution over the two income growth regimes is $\left(\frac{\lambda_2}{\lambda_1+\lambda_2}, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$.

The agent can invest in both a risk-free asset with a constant interest rate r and a risky asset (i.e., the market portfolio) with a risky return r_t^e. The instantaneous return dr_t^e of the risky market portfolio over dt is given by

$$dr_t^e = (r + \pi)\, dt + \sigma_e dB_{e,t}, \tag{2}$$

where π is the market risk premium, σ_e is the standard deviation of the market return, and $B_{e,t}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$ and is correlated with the Brownian motion, $B_{y,t}$. Let ρ_{ye} be the contemporaneous correlation between the labor income process and the return of the risky asset. When $\rho_{ye} = 0$, the labor income risk is idiosyncratic and is uncorrelated with the risky market return; when $\rho_{ye} = 1$, the labor income risk is perfectly correlated with the risky

[10] If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. The $\rho = 0$ case corresponds to a simple Brownian motion with drift. The larger ρ is, the less y tends to drift away from $\overline{y}$. As ρ goes to ∞, the variance of y goes to 0, which means that y can never deviate from $\overline{y}$.

market return. Using the Cholesky decomposition, the labor income process can be rewritten as:

$$dy_t = (\mu(Z_t) - \rho y_t)\,dt + \rho_{ye}\sigma_y dB_{e,t} + \sqrt{1-\rho_{ye}^2}\sigma_y dB_{i,t}, \tag{3}$$

where $B_{i,t}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$ and is independent of $B_{e,t}$. The consumer's financial wealth evolution is given by

$$dw_t = (rw_t + y_t - c_t)\,dt + \alpha_t(\pi dt + \sigma_e dB_{e,t}), \tag{4}$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t.

3.2 Incomplete Information about Labor Income

Following Wang (2009), in this paper we assume that the parameter governing income growth, $\mu(Z_t)$, is *unknown* to the investors. If the investor does not know the income growth parameter, $\mu(Z_t)$, he or she has to form a belief about the value of the income growth parameter by observing the realized labor income y.[11] We denote the augmented filtration generated by y_t and B_t as $\{\mathcal{F}_t^y, t \geq 0\}$, where $\mathcal{F}_t^y = \sigma(y_s, B_s, s \leq t)$. The available information set $\mathcal{F}_t^y$ represents the information contained in the past paths of y_t and B_t, but does not include the true values of Z_t or $\mu(Z_t)$. Specifically, we use p_t to denote his or her time-t belief that the income growth is high, i.e., $p_t = \Pr(\mu = \mu_1 | \mathcal{F}_t^y)$. Let $\overline{\mu}$ denote the conditional expectation of income growth, μ, with respect to the incomplete information filtration $\{\mathcal{F}_t^y, t \geq 0\}$, and it can be written as:

$$\overline{\mu} = p_t\mu_1 + (1-p_t)\,\mu_2 = \mu_2 + \delta p_t,$$

where $\delta = \mu_1 - \mu_2 > 0$. The corresponding conditional variance is:

$$\mathrm{var}(\mu(Z_t)\ |\mathcal{F}_t^y) = \delta^2 p_t(1-p_t).$$

During $(t, t+dt)$, the expected change in $dy_t \equiv y_{t+dt} - y_t$ is $(\overline{\mu} - \rho y_t)\,dt$ and the corresponding unanticipated change is $dy_t - (\overline{\mu} - \rho y_t)\,dt$. Then the unanticipated fluctuation $d\hat{B}_{it}$ of labor income y_t can be written as:

$$d\hat{B}_{i,t} = dB_{i,t} + \frac{1}{\sqrt{1-\rho_{ye}^2}\sigma_y}(\mu - \overline{\mu})\,dt, \tag{5}$$

where $\hat{B}_{i,t}$ is a new standard Brownian motion with respect to $\mathcal{F}_t^y$ and is independent of $B_{e,t}$. Here $\hat{B}_{it}$ serves as the innovation process for belief updating. Combining (3) with (5), we can

[11] Sampson (1994) adopted the same learning mechanism and examined how the uncertainty about the income growth parameter affects aggregate wealth accumulation in a discrete-time CARA precautionary saving model without portfolio choice.

rewrite the income process as follows:

$$dy_t = (\overline{\mu} - \rho y_t)\, dt + \rho_{ye}\sigma_y dB_{e,t} + \sqrt{1-\rho_{ye}^2}\sigma_y d\hat{B}_{i,t}. \tag{6}$$

Finally, the belief process can be written as follows:[12]

$$dp_t = [\lambda_2 - (\lambda_1 + \lambda_2)\, p_t]\, dt + \sigma_y^{-1}\delta p_t\,(1-p_t)\left(\rho_{ye} dB_{e,t} + \sqrt{1-\rho_{ye}^2} d\hat{B}_{i,t}\right). \tag{7}$$

It is worth noting that when $\lambda_1 = \lambda_2 = 0$, the above belief updating process reduces to the case with unknown and *constant* income growth (i.e., Model II of Wang 2009) and (7) reduces to the following martingale process:

$$dp_t = \sigma_y^{-1}\delta p_t\,(1-p_t)\left(\rho_{ye} dB_{e,t} + \sqrt{1-\rho_{ye}^2} d\hat{B}_{i,t}\right). \tag{8}$$

3.3 Recursive Utility

In this paper, we assume that agents in our model economy have recursive preferences of the Kreps-Porteus/Epstein-Zin type, and can disentangle the degree of risk aversion from the elasticity of intertemporal substitution.[13] Specifically, for every stochastic consumption-portfolio stream, $\{c_t, \alpha_t\}_{t=0}^{\infty}$, the utility stream, $\{f(U_t)\}_{t=0}^{\infty}$, is recursively defined as follows: For every stochastic stream, $\{c_t, \alpha_t\}_{t=0}^{\infty}$, the utility stream, $\{f(U_t)\}_{t=0}^{\infty}$, is recursively defined as follows:

$$V(U_t) = \left(1 - e^{-\beta\Delta t}\right) V(c_t) + e^{-\beta\Delta t} V\left(CE_t\left[U_{t+\Delta t}\right]\right), \tag{9}$$

where Δt is time interval, $\beta > 0$ is the agent's subjective discount rate, $V(c_t) = (-\psi)\exp(-c_t/\psi)$, $V(U_t) = (-\psi)\exp(-U_t/\psi)$,

$$CE_t\left[U_{t+\Delta t}\right] = G^{-1}\left(E_t\left[G\left(U_{t+\Delta t}\right)\right]\right), \tag{10}$$

is the certainty equivalent of U_{t+1} conditional on the period t information, and $G(U_{t+\Delta t}) = -\exp(-\gamma U_{t+\Delta t})/\gamma$. In (9), $\psi > 0$ governs the elasticity of intertemporal substitution (EIS), while $\gamma > 0$ governs the coefficient of absolute risk aversion (CARA).[14] In other words, a high value of ψ corresponds to a strong willingness to substitute consumption over time, and a high value of γ implies a high degree of risk aversion. Note that when $\psi = 1/\gamma$, the functions V and G are the same and the recursive utility reduces to the standard time-separable expected utility function used in Caballero (1990) and Wang (2003, 2009). In addition, $\psi = 1/\gamma$ also implies that the consumer is indifferent about the time at which uncertainty is resolved.[15]

[12] Note that in this model the income process and the belief process are perfectly correlated.

[13] Although the expected utility model has many attractive features, it implies that the agent's elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aversion. However, conceptually risk aversion and intertemporal substitution capture two distinct aspects of decision-making problem.

[14] It is well-known that the CARA utility specification is tractable for deriving the consumption function or optimal consumption-portfolio rules in different settings. See Merton (1971), Caballero (1990), Calvet (2001), Wang (2003, 2009), Angeletos and Calvet (2006), and Luo (2016).

[15] Note that the consumer prefers early resolution of uncertainty if $\gamma > 1/\psi$ and prefers late resolution if $\gamma < 1/\psi$.

In the standard full-information rational expectations (FI-RE) model, the typical investor maximizes (9), subject to (4), (6), and (7). Before we introduce model uncertainty due to robustness, we first write down the Hamilton-Jacobi-Bellman (HJB) equation when the typical investor trusts the model since it can help facilitate the introduction of model uncertainty in the next section:

$$\beta V(J_t) = \sup_{\{c_t, \alpha_t\}} \{\beta V(c_t) + \mathcal{D}V(J_t)\}, \tag{11}$$

where $J_t = -\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_1 f(p_t)$,[16]

$$\mathcal{D}V(J_t) = V'(J_t)\left((\partial J)^T \cdot E[ds_t] - \frac{\gamma}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right]\right),$$

$s_t = \begin{bmatrix} w_t & y_t & f(p_t) \end{bmatrix}^T$, $ds_t = \begin{bmatrix} dw_t & dy_t & df(p_t) \end{bmatrix}^T$, $\partial J = \begin{bmatrix} J_w & J_y & J_f \end{bmatrix}^T$,

$$\Sigma = \begin{bmatrix} \alpha_t^2\sigma_e^2 & \rho_{ye}\sigma_y\alpha_t\sigma_e & \rho_{ye}\alpha_t\sigma_e\sigma_y^{-1}\delta p_t(1-p_t)f'(p_t) \\ \rho_{ye}\sigma_y\alpha_t\sigma_e & \sigma_y^2 & \delta p_t(1-p_t)f'(p_t) \\ \rho_{ye}\alpha_t\sigma_e\sigma_y^{-1}\delta p_t(1-p_t)f'(p_t) & \delta p_t(1-p_t)f'(p_t) & \sigma_y^{-2}\left[\delta p_t(1-p_t)f'(p_t)\right]^2 \end{bmatrix}. \tag{12}$$

$f(p_t)$ satisfies the following nonlinear ODE:

$$\begin{aligned} rf(p_t) = {} & \frac{\delta}{r+\rho}p_t - \left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\gamma}{r+\rho}\left(1-\rho_{ye}^2\right)\right] f'(p_t)\,\delta p_t(1-p_t) + \left[\lambda_2 - (\lambda_1+\lambda_2)p_t\right] f'(p_t) \\ & - \frac{r\gamma}{2\sigma_y^2}\left(1-\rho_{ye}^2\right)\left[f'(p_t)\,\delta p_t(1-p_t)\right]^2 + \frac{1}{2\sigma_y^2}f''(p_t)\left(\delta p_t(1-p_t)\right)^2. \end{aligned} \tag{13}$$

and the transversality condition, $\lim_{t\to\infty} E\left|\exp(-\beta t)V_t\right| = 0$ holds at the optimum. (See Appendix 7.1 for the derivation.)

3.4 Incorporating Model Uncertainty due to Robustness

As argued in Hansen and Sargent (2007), the simplest version of robustness considers the question of how to make optimal decisions when the decision-maker does not know the true probability model that generates the data. The main goal of introducing robustness is to design optimal policies that not only work well when the reference (or approximating) model governing the evolution of the state variables is the true model, but also performs reasonably well when there is some type of model misspecification. To introduce robustness into our model proposed above, we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004) to assume that consumers are concerned about the model misspecifications and take Equations (4), (6), and (7) as the approximating

[16] Here α_0, α_1, and α_2 are undetermined coefficients.

model.[17] The corresponding distorting model can thus be obtained by adding an endogenous distortion $\upsilon(s_t)$ to the approximating model:

$$ds_t = (\Lambda + \Sigma \cdot \upsilon_t)\, dt + \sigma \cdot dB_t, \tag{14}$$

where $\upsilon_t = \begin{bmatrix} \upsilon_{1,t} & \upsilon_{2,t} & \upsilon_{3,t} \end{bmatrix}^T$, $\Lambda = \begin{bmatrix} rw_t + y_t - c_t + \alpha_t \pi & \mu_2 + \delta p_t - \rho y_t & \lambda_2 - (\lambda_1 + \lambda_2)\, p_t \end{bmatrix}^T$

$$\sigma \cdot dB_t = \begin{bmatrix} \alpha_t \sigma_e dB_{e,t} \\ \rho_{ye}\sigma_y dB_{e,t} + \sqrt{1-\rho_{ye}^2}\sigma_y d\hat{B}_{i,t} \\ \sigma_y^{-1}\delta p_t (1-p_t) f'(p_t) \left(\rho_{ye} dB_{e,t} + \sqrt{1-\rho_{ye}^2} d\hat{B}_{i,t}\right) \end{bmatrix},$$

and $\Sigma \equiv \sigma\sigma^T$ is given in (12).

Under RB, the HJB can be thus written as

$$\beta V(J_t) = \sup_{\{c_t, \alpha_t\}} \inf_{\upsilon_t} \left\{ \beta V(c_t) + \mathcal{D}V(J_t) + \frac{1}{\vartheta_t}\mathcal{H} \right\}, \tag{15}$$

where

$$\mathcal{D}V(J_t) = V'(J_t)\left((\partial J)^T \cdot E[ds_t] + (\partial J)^T \cdot \Sigma \cdot \upsilon_t - \frac{\gamma}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J \right] \right), \tag{16}$$

the first two terms in (16) are the expected continuation payoff when the state variable follows (14), i.e., the alternative model based on drift distortion $\upsilon(s_t)$, $\mathcal{H} = \left(\upsilon_t^T \cdot \Sigma \cdot \upsilon_t\right)/2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models, and $1/\vartheta_t$ is the weight on the entropy penalty term.[18] More specifically, $(\partial J)^T \cdot \Sigma \cdot \upsilon_t$ is the adjustment to the expected continuation value when the state dynamics is governed by the distorted model with the mean distortion υ_t. The final term, $\mathcal{H}/\vartheta_t$, in (15) quantifies the penalty due to RB. As shown in AHS (2003), the objective $\mathcal{D}V$ defined in (16) plays a crucial role in introducing robustness. A key insight of AHS (2003) is that this differential expectations operator reflects a particular underlying model for the state variable. The consumer accepts the approximating model as the best approximating model, but is still concerned that it is misspecified. He or she therefore wants to consider a range of models (i.e., the distorted model, (14)) surrounding the approximating model when computing the continuation payoff. A preference for robustness is then achieved by having the agent guard against the distorting model that is reasonably close to the approximating model.

[17] As argued in Hansen and Sargent (2007), the agent's commitment technology is irrelevant under RB if the evolution of the state is backward-looking. We therefore do not specify the commitment technology of the consumer in the RB models of this paper.

[18] The last term in (17) is due to the investor's preference for robustness. Note that the $\vartheta_t = 0$ case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty)* robust control problem.

The drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (14) and (ii) an entropy penalty:

$$\inf_{\upsilon}\left[V'(J_t)(\partial J)^T\cdot\Sigma\cdot\upsilon_t+\frac{1}{\vartheta_t}\mathcal{H}\right], \tag{17}$$

where ϑ_t is fixed and state-independent in AHS (2003); whereas it is state-dependent in Maenhout (2004, 2006). The key reason for using a state-dependent counterpart ϑ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem with the CRRA utility function.[19] In this paper, we also specify that ϑ_t is state-dependent ($\vartheta(s_t)$) in the CARA-Gaussian setting. The main reason for this specification is to guarantee homotheticity, which keeps robustness from diminishing as the value of the total wealth increases.[20] Note that the evil agent's minimization problem, (17), becomes invariant to the scale of total resource, s_t when using the state-dependent specification of ϑ_t. Solving first for the infimization part of (17) yields:

$$\upsilon_t^*=-\vartheta_t V'(J_t)\,\partial J \tag{18}$$

where $\vartheta_t=-\vartheta/V(J_t)>0$ and ϑ is a constant (see Appendix 7.3 for the derivation). Following Uppal and Wang (2003) and Liu, Pan, and Wang (2005), here we can also define "$1/V(J_t)$" in the ϑ_t specification as a *normalization* factor that is introduced to convert the relative entropy (i.e., the distance between the approximating model and the distorted model) to units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model. It is worth noting that adopting a slightly more general specification, $\vartheta_t=-\varphi\vartheta/V(J_t)$ where φ is a constant, does not affect the main results of the paper. The reason is as follows. We can simply define a new constant, $\widetilde{\vartheta}=\varphi\vartheta$, and $\widetilde{\vartheta}$, rather than ϑ, will enter the decision rules. It is worth noting that the state-dependent RB specification, ϑ_t, is similar to the AR(1) ambiguity shocks proposed in Bhandari, Borovička, and Ho (2016). They identified an AR(1) ambiguity shock using survey data from the Surveys of Consumers and the Survey of Professional Forecasters, and found that in the data, the ambiguity shock is an important source of labor market fluctuations.

Using a given detection error probability, we can easily calibrate the corresponding value of $\widetilde{\vartheta}$ that affects the optimal consumption-portfolio rules.[21] Substituting for υ^* in (15) leads to the following HJB:

$$\beta V(J_t)=\sup_{\{c_t,\alpha_t\}}\left\{\beta V(c_t)+V'(J_t)\left((\partial J)^T\cdot E[ds_t]-\frac{1}{2}\widetilde{\gamma}\left[(\partial J)^T\cdot\Sigma\cdot\partial J\right]\right)\right\}, \tag{19}$$

where $\widetilde{\gamma}=\gamma+\vartheta/\psi$.

[19] See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness".

[20] Note that the impact of robustness wears off if we assume that ϑ_t is constant. This is clear from the procedure of solving the robust HJB proposed. (See Appendix 7.3 for the details.)

[21] See Section **??** for the detailed procedure to calibrate the value of ϑ using the detection error probabilities.

4 Main Results and Implications

This section presents main results. We first explain our theoretical results, followed by quantitative results based on a calibration of the key RB parameter. We then test our model implications using data and quantitatively address welfare implications.

4.1 Theoretical Implications for Robustly Strategic Consumption-Asset Allocation

Since the labor income risk is partially hedged by the risky asset, the discount rate of the labor income flow is no longer the risk-free rate r for risk-neutral agents. Instead, given that the Sharpe ratio of the market portfolio is π/σ_e and the risk-free interest rate is r, there exists a unique stochastic discount factor ζ_t in financial market satisfying:

$$d\zeta_t = -\zeta_t \left(rdt + \frac{\pi}{\sigma_e} dB_{e,t} \right),$$

where $\zeta_0 = 1$. As in Koo (1998) and Munk and Sørensen (2010), the present value of the expected stream of future labor income under incomplete information can be expressed as

$$h(y_t, p_t) = E_t^{\mathbb{P}} \left[\int_t^{\infty} \frac{\zeta_s}{\zeta_t} y_s ds \middle| \mathcal{F}_t^y \right] = E_t^{\mathbb{Q}} \left[\int_t^{\infty} e^{-r(s-t)} y_s ds \middle| \mathcal{F}_t^y \right], \tag{20}$$

where $\mathbb{Q}$ is the risk-neutral probability measure with respect to $\mathbb{P}$.

Proposition 1 *The human wealth under incomplete information can be decomposed as follows:*

$$h(y, p) = m(y) + n(p), \tag{21}$$

where (i) the y component, $m(y)$, has the following form

$$m(y) = \frac{1}{r+\rho} \left[y + \frac{1}{r} \left(\mu_2 - \frac{\rho_{ye}\sigma_y \pi}{\sigma_e} \right) \right], \tag{22}$$

and (ii) the p component, $n(p)$, is the solution of the following differential equation

$$rn(p) = \frac{\delta p}{r+\rho} + \left[\lambda_2 - (\lambda_1 + \lambda_2) p - \frac{\rho_{ye}\pi}{\sigma_e \sigma_y} \delta p(1-p) \right] n'(p) + \frac{1}{2\sigma_y^2} \delta^2 p^2 (1-p)^2 n''(p), \tag{23}$$

subject to the boundary conditions: $rn(0) = \lambda_2 n'(0)$ and $rn(1) = \frac{\delta}{r+\rho} - \lambda_1 n'(1)$.

Proof. See Appendix 7.2. ■

Following the standard procedure, we can then solve (19) and obtain the robustly strategic consumption-portfolio rules. The following proposition summarizes the solution:

Proposition 2 *With unknown income growth, the robust consumption, precautionary saving, and portfolio rules are*

$$c_t^* = r\left(w_t + g\left(y_t\right) + f\left(p_t\right)\right) + \frac{\left(\beta - r\right)\psi}{r} + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}, \tag{24}$$

$$\Gamma \equiv \Gamma_y + \Gamma_p, \tag{25}$$

$$\alpha = \alpha_s + \alpha_y + \alpha_p, \tag{26}$$

where $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$ is the effective coefficient of absolute risk aversion,

$$g\left(y_t\right) = \frac{1}{r+\rho}\left[y_t + \frac{1}{r}\left(\mu_2 - \frac{r\widetilde{\gamma}\sigma_y^2\left(1-\rho_{ye}^2\right)}{2\left(r+\rho\right)} - \frac{\rho_{ye}\sigma_y\pi}{\sigma_e}\right)\right] \tag{27}$$

is the agent's certainty equivalent human wealth when his income growth rate is known to be low ($\mu = \mu_2$), $f\left(p_t\right)$ is the certainty equivalent wealth of learning,

$$\Gamma_y \equiv r\left[m(y_t) - g\left(y_t\right)\right] = \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\sigma_y^2}{2\left(r+\rho\right)^2}, \tag{28}$$

$$\begin{aligned}\Gamma_p &\equiv r\left[n(p_t) - f\left(p_t\right)\right] \\ &= -\frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{\left(r+\rho\right)^2\left(r+\lambda_1+\lambda_2\right)\left(r+2\left(\lambda_1+\lambda_2\right)\right)}\left(p_t^2 - \frac{r+2\lambda_1}{r+\lambda_1+\lambda_2}p_t - \frac{r+2\lambda_1}{r+\lambda_1+\lambda_2}\frac{\lambda_2}{r}\right)\end{aligned} \tag{29}$$

are the standard precautionary saving demand and the learning-induced precautionary saving demand, respectively,

$$\alpha_s \equiv \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} \tag{30}$$

is the standard speculation demand,

$$\alpha_y \equiv -\frac{\rho_{ye}\sigma_y}{\sigma_e\left(r+\rho\right)} \tag{31}$$

is the labor income-hedging demand,

$$\alpha_p \equiv -\frac{\rho_{ye}}{\sigma_e\sigma_y}\delta p_t\left(1-p_t\right)f'\left(p_t\right) \tag{32}$$

is the learning-induced hedging demand, and $f\left(p_t\right)$ solves the following non-linear ODE:

$$\begin{aligned}rf\left(p_t\right) = \frac{\delta}{r+\rho}p_t - \left[\frac{\pi\rho_{ye}}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{r+\rho}\right]\delta p_t\left(1-p_t\right)f'\left(p_t\right) + \left[\lambda_2 - \left(\lambda_1+\lambda_2\right)p_t\right]f'\left(p_t\right) \\ + \frac{1}{2\sigma_y^2}\left[\delta p_t\left(1-p_t\right)\right]^2 f''\left(p_t\right) - \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{2\sigma_y^2}\left[\delta p_t\left(1-p_t\right)f'\left(p_t\right)\right]^2.\end{aligned} \tag{33}$$

subject to the boundary conditions: $rf\left(0\right) = \lambda_2 f'\left(0\right)$ and $rf\left(1\right) = \frac{\delta}{r+\rho} - \lambda_1 f'\left(1\right)$, for $p \in [0,1]$. Furthermore, the transversality condition (TVC), $\lim_{t\to\infty} E\left|\exp\left(-\delta t\right)J_t\right| = 0$, holds at optimum. Finally, the value function can be written as

$$V(w_t, y_t, p_t) = -\frac{\beta\psi}{r}\exp\left\{-\frac{r}{\psi}\left[w_t + g\left(y_t\right) + f\left(p_t\right) + \frac{\left(\delta - r\right)\psi}{r^2} + \frac{\pi^2}{2r^2\widetilde{\gamma}\sigma_e^2}\right]\right\}. \tag{34}$$

Proof. See Appendix 7.3. ■

Expression (24) clearly shows that the consumption function can be decomposed into four components: (i) the annuity value of financial wealth, rw_t, (ii) the annuity value of the risk-adjusted and robustness-adjusted certainty equivalent human wealth under incomplete information, $r\left(g\left(y_t\right)+f\left(p_t\right)\right)$, (iii) the effect of the relative impatience measured by $\psi\left(\beta-r\right)/r$, and (iv) the wealth effect of investing in the risky asset, $\pi^2/\left(2\widetilde{\gamma}\sigma_e^2\right)$.

Furthermore, the certainty equivalent human wealth contains two terms: one is the risk-adjusted and robustness-adjusted value of labor income, $g\left(y_t\right)$, and the other is the value of learning under robustness, $f(p_t)$, which is due to the RB agent's belief updating about his unobservable income growth. Unlike the incomplete information model of Wang (2009), we can see from (27) and (33) both the adjusted-certainty equivalent human wealth $g\left(y_t\right)$ and the value of learning $f\left(p_t\right)$ in our model are affected by the preference for robustness. From (27), it is straightforward to show that $g\left(y_t\right)$ decreases with the degree of robustness measured by ϑ because the effective coefficient of absolute aversion $(\widetilde{\gamma})$ increases with ϑ. The higher the degree of robustness, the lower the risk-adjusted and robustness-adjusted discounted present value of labor income. Given the complexity of (33), we cannot explicitly inspect how RB affects the value of learning, $f(p_t)$. In the next section, we will quantitatively explore the effects of RB on $f(p_t)$ after calibrating and estimating the parameter values.

From (25), (28), and (29), it is clear that under incomplete information, we can decompose the precautionary saving demand (Γ) into the standard and learning-induced components. The standard component (Γ_y) is mainly determined by the interaction of risk aversion, robustness, and the stochastic properties of labor income including labor income uncertainty (σ_y^2) and the correlation between labor income and the equity return (ρ_{ye}), whereas the learning-induced component (Γ_p) is mainly determined by the interaction of risk aversion, robustness, incomplete information about income growth, and the stochastic properties of both the equity return and labor income including the volatility of labor income, the equity return, and their correlation.[22]

From (26), we can see that the total demand for the risky asset contains three components: (i) the standard speculation demand (α_s), (ii) the labor income-hedging demand (α_y), and (iii) the learning-induced hedging demand (α_p). Expressions (30) and (31) show that RB reduces the traditional speculation demand (α_s) by a factor, $1+\vartheta$, but does not affect the income-hedging demand of the risky asset (α_y). In other words, RB increases the relative importance of the income hedging demand to the speculation demand by increasing the effective coefficient of absolute risk aversion $(\widetilde{\gamma})$.

From (32), we can see that the learning-induced hedging demand (α_p) is determined by both the brief p and the marginal value of learning $f'\left(p\right)$. It is also clear from (33) that when the

[22] RB can directly affect Γ_p indirectly via its impact on $f\left(p\right)$.

equity return and labor income are perfectly correlated (i.e., $\rho_{ye} = \pm 1$), the impact of RB on $f(p_t)$ and the learning-induced hedging demand disappears because $\widetilde{\gamma}\left(1-\rho_{ye}^2\right) = 0$ in this case. We will explore the quantitative implication of learning and robustness on α_p in the next section after solving Equation (33) numerically.

4.2 Calibration and Quantitative Findings

In this subsection, to fully explore how RB affects the joint behavior of portfolio choice, consumption, and labor income when the mean of income growth is unknown, we first describe how we calibrate the value of the RB parameter (ϑ) that governs the degree of robustness, and then present quantitative results on how the interaction of learning and RB affects the precautionary saving demand and the portfolio choice.

We adopt the calibration procedure outlined in AHS (2003) and Maenhout (2004) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by q is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the investor to distinguish the two models (see Online Appendix A for the detailed calibration procedure using the value of q).

Using the data set documented in Campbell (2003), we set the parameter values for asset returns and volatility, and consumption as follows: $\pi = 0.05$, $r = 0.02$, and $\sigma_e = 0.156$. For the labor income process, we follow Wang (2009) and Luo, Nie, Wang and Young (2016), and set that $\lambda_1 = \lambda_2 = 0.03$, $\delta = \mu_1 - \mu_2 = 0.05$, $\rho = 0.0834$, and $\sigma_y = 0.182$. The magnitude of the EIS (ψ) is a key issue in macroeconomics and asset pricing. For example, Parker (2002) and Vissing-Jorgensen and Attanasio (2003) estimate the IES to be well in excess of one. Hall (1988) and Campbell (2003), on the other hand, estimate its value to be well below one. Here we choose $\psi = 0.3$, 0.5, and 0.8 for illustrative purposes.[23] Figure 1 illustrates how DEP (q) varies with the value of ϑ for different values of ψ, γ, p, and ρ_{ye}. We can see from the left upper panel of the figure that the stronger the preference for robustness (higher ϑ), the less the value of q is, holding other parameters constant. For example, when $\gamma = 3$ and $\psi = 0.5$, $q = 35\%$ when $\vartheta = 1$, while $q = 18\%$ when $\vartheta = 4$. Both values of q are reasonable as argued in AHS (2002), Maenhout

[23] Guvenen (2006) finds that stockholders have a higher EIS (around 1.0) than non-stockholders (around 0.1). Crump, Eusepi, Tambalotti, and Topa (2015) find that the EIS is precisely and robustly estimated to be around 0.8 in the general population using the newly released FRBNY Survey of Consumer Expectations (SCE).

(2004), and Hansen and Sargent (Chapter 9, 2007). We set the value of ϑ to be in the range of $[1, 4]$ in our subsequent quantitative analysis. In addition, we can also see from this panel that given the same value of ϑ, q increases with the value of ψ, and ψ has significant impact on the relationship between ϑ and q. For example, q is about 20% when $\psi = 0.5$, while it is about 10% when $\psi = 0.3$. We can see from the right upper panel of Figure 1 that the degree of risk aversion has no impact on the mapping between ϑ and q. In other words, the degree of risk aversion is irrelevant for calibrating ϑ using DEP, q. From the lower panels of this figure, it is clear that p and ρ_{ye} only have trivial impact on the relationship between ϑ and q.

Figure 2 illustrates how the interactions of the brief (p) with the degree of robustness (ϑ), intertemporal substitution (ψ), risk aversion (γ), and the correlation between the risky asset and labor income (ρ_{ye}) affect the learning value ($n(p_t)$ or $f(p_t)$). It is clear from the figure that the value of learning is increasing with the probability that the consumer believes that the current state is good for given values of ϑ, ψ, γ, and ρ_{ye}.[24] The result is consistent with that obtained in Wang (2009)'s expected utility model without the presence for robustness. In addition, since the consumer is risk- and uncertainty-averse, the risk- and robustness-neutral learning value, $n(p_t)$, is higher than th learning value, $f(p_t)$ obtained in our RB model, and $f(p_t)$ is decreasing with the value of ϑ and γ for given p. Furthermore, from the right upper panel of the figure, we can see that for given p, $f(p_t)$ is increasing with ψ and is decreasing with ρ_{ye}. The intuition for this result is that the lower the EIS, the larger the amplification effect on RB. In addition, we can see from the right lower panel of the figure that $f(p_t)$ decreases with the degree of the learning-induced hedging motive due to $\rho_{ye} \neq 0$.

Figure 3 shows that both the standard and learning-induced precautionary saving components (Γ_y and Γ_p) are increasing with ϑ, whereas the relative importance of the learning-induced component is independent of ϑ.[25] Furthermore, we can see from this figure that learning-induced precautionary savings demand under RB is concave in brief p. Specifically, when the consumer is more uncertain about the current situation (i.e., the value of p deviates more from 0 or 1, and is in the interior region of p), this component is higher. It is worth noting that Γ_p is not symmetric around $p = 0.5$, and is skewed to the left. This is due to the fact that both $n(p_t)$ and $f(p_t)$ are slightly convex.[26] Figure 4 shows that both Γ_y and Γ_p are decreasing with ψ. The intuition is simple: The lower the value of ψ, the greater its contribution to the effective coefficient of risk aversion (ϑ/ψ).

[24] The evidence on the value of the correlation between the equity return and labor income is mixed. Following Campbell and Viceira (2002), here we set $\rho_{ye} = 0.35$.

[25] We can also show that RB has no impact on the relative importance of the learning-induced precautionary saving by inspecting (28) and (29). The terms with $\dot{\vartheta}$ in these two expressions are just cancelled out.

[26] It is straightforward to show if we solve the ODE, (33), approximately when δ is small. See Appendix 7.3 for the detailed proof.

To separate the different effects of incomplete information and robustness on Γ_p, we further decompose Γ_p as $\Gamma_p = \Gamma_{p,1} + \Gamma_{p,2}$, where $\Gamma_{p,1}$ is the learning-induced hedging demand when there is no preference for robustness (i.e., $\vartheta = 0$), and $\Gamma_{p,2}$ capture the additional learning-induced hedging demand due to robustness. Figure 5 clearly shows that $\Gamma_{p,2}$ increases with the value of ϑ and γ, whereas it decreases with the value of ψ, for given values of p. The interaction of CARA, EIS, and RB plays an important role in determining the amount of learning-induced precautionary savings. For example, when $\vartheta = 1$, $\gamma = 3$, and $\psi = 0.5$, the two components are equally important in determining the learning-induced precautionary saving demand. When ϑ is increased to 2.5, $\Gamma_{p,2}$ is about 2.7 times larger than $\Gamma_{p,1}$.

Figure 6 illustrates how RB affects the total demand for the risky asset and the learning-induced hedging demand (α_p). It is clear from the upper panels of the figure that α is decreasing with ϑ and learning does not have significant impact on α_p because ϑ does not appear in the α_p expression explicitly and affects α_p via the $f'(p)$ term. From the lower panels of the figure, we can also see that α_p accounts for a significant fraction of the total demand for the risky asset. For example, when $\vartheta = 4$, $\psi = 0.5$, and $p = 0.65$, $|\alpha_p|/\alpha$ is about 26.5%. Figure 4 shows that α is increasing with ψ and learning does not have significant impact on α_p. The reason is the same as before: The lower the value of ψ, the greater its contribution to the effective coefficient of risk aversion (ϑ/ψ). In addition, we can also see that learning-induced demand for the risky asset under RB is concave in brief p. Specifically, this demand is higher for the more uncertainty-averse agent when p is greater than 0.5. It is also clear from Figures 6 and 7 that α_p is not symmetric around $p = 0.5$, and is slightly skewed to the right. This is due to the fact that $f(p_t)$ is slightly convex and the fact that $f'(p_t)$ appears in Expression (32).

To separate the different effects of incomplete information and ambiguity on α_p, we further decompose α_p as $\alpha_p = \alpha_{p,1} + \alpha_{p,2}$, where $\alpha_{p,1}$ is the learning-induced hedging demand when there is no preference for robustness (i.e., $\vartheta = 0$), and $\alpha_{p,2}$ capture the additional learning-induced hedging demand due to robustness. Figure 8 clearly shows that the relative importance of RB in learning-induced portfolio choice, $\alpha_{p,2}/\alpha_{p,1}$, increases with the value of p for given values of ϑ. We can also see from the figure that when $p = 0.5$, $\alpha_{p,2}/\alpha_{p,1} = 0$. The reason behind this result is that $\alpha_{p,2}$ depends on the value of $f_2'(p_t)$ and $f_2'(p_t) = 0$ when $p = 0.5$, where $f_2(p_t)$ is part of $f(p_t)$: $f(p_t) \approx f_1(p_t)\delta + f_2(p_t)\delta^2$.[27] In addition, for given values of p, the ratio, $\alpha_{p,2}/\alpha_{p,1}$, increases with ϑ when p is greater than 0.5, whereas it decreases with ϑ when p is less than 0.5.

[27] See Appendix 7.3 for the derivation of $f_2(p_t)$.

4.3 Testable Implications of Ignorance

Our model has some interesting testable implications. The theoretical and quantitative results in the previous sections suggest that more ignorance leads to less holding of risky assets. This can be tested in two steps. First, there is evidence that highly educated people are usually more financial literate (i.e., less ignorant). For example, Mitchell and Lusardi (2014) provided a recent survey testing participants' command of financial principles and planning by using their three-question (about compounding, inflation and risk) poll. They found that people with more education did better. In the U.S., for example, 44.3% of those with college degrees answered all three questions correctly, compared with 31.3% for those with some college, 19.2% of those with only a high school degree and 12.6% for those with less than a high school degree. Among those with post-graduate degrees, 63.8% got all answers right. Other countries showed similar results. This suggests that people with higher education probably have better knowledge about the structure and parameters of the model, and are thus less ignorant about the model specification and parameter uncertainty.

Second, we test if the education level is positively correlated with risky-asset holdings. To do this, we use data from the Panel Study of Income Dynamics (PSID) which contains information about individual households' wealth, income, and stock holdings. In particular, we divide households into three groups by their educational levels, and then examine their holdings of stocks, in both absolute terms and relative terms. The details of the data set are explained in Appendix 7.4. The number of households in each category is reported in Table 1 and the main findings are summarized in Tables 2 and 3. Specifically, Table 2 shows that the amount of stock holdings increase with the educational level, not only for the whole sample, but also by income groups.[28] To further control the effects of education on income which also influences the absolute level of stockholding, we report the relative stockholding, defined as the share of stockholding in households' total wealth, in Table 3. In Table 3, when calculating households' wealth, we consider two cases: one includes home equities and one excludes home equities. It is clear from the table that the share of wealth invested in stocks is positively correlated with the household's educational level in both cases. These findings can therefore be consistent with our model's predictions and highlight the importance of ignorance-induced uncertainty in explaining the data. Specifically, less well-educated investors probably face greater ignorance-induced uncertainty; consequently, they rationally choose to invest less in the stock market even if the correlation between their labor income and equity returns is the same as that for the well-educated investors.[29]

[28] This is consistent with earlier work in the literature. For example, Tables 1 and 2 in Luo (2016) show a positive relationship between the mean value of stockholding and the education level at all income and net worth levels using the Survey of Consumer Finances (SCF) data. Haliassos and Bertaut (1995) also found that the share invested in the stock market is substantially larger among those with at least a college degree compared to those with less than high school education at all income levels.

[29] As documented in Campbell (2006), there is some evidence that households understand their own limitations

To quantitatively show how our model has the potential to reconcile with the empirical evidence, Table 4 provides a numerical example in which the model matches exactly the relative risk-asset holdings in the data under reasonable parameter values. First, for parameters that are not related to MU and PU, we use the same values as in the previous subsection: $\pi = 0.05$, $r = 0.02$, $\sigma_e = 0.156$, $\lambda_1 = \lambda_2 = 0.03$, $\delta = 0.05$, $\rho = 0.0834$, and $\sigma_y = 0.182$. Then, as reported in the upper panel in Table 4, we set $\vartheta = 3.92$ and $p = 0.5$ for the least educated group (i.e., people not finishing high school face significant model and parameter uncertainty), $\vartheta = 3.3$ and $p = 0.5$ for the group with high school and some college education, and $\vartheta = 1$ and $p = 1$ for the group with college and above education. Using (26), we can easily calculate that the optimal amount invested in the risky asset (α) for these three groups. As the bottom panel in Table 4 shows, the ratio of the risky assets of the middle educated group to that of the less educated group is 2.76 in the model, while the ratio of the risky assets of the well educated group to that of the less educated group is 21.4 in the model, both matching exactly the empirical counterparts in the whole sample.

To further illustrate the importance of the interaction between MU and PU in driving the key results, Figure 14 shows how the relative risky-asset holding varies with the degree of PU under different assumptions of MU. Specifically, the blue line shows the ratio of risky assets for the group with $\vartheta = 1$ (i.e., the group facing relatively less MU) to the group with $\vartheta = 2$ (i.e., the group facing slightly larger MU); the red thin dashed line shows the ratio of risky assets for the group with $\vartheta = 1$ to the group with $\vartheta = 3.5$; and the red thick dashed line shows the ratio of risky assets for the group with $\vartheta = 1$ to the group with $\vartheta = 4$. In other words, these three lines are comparing three different groups of investors who face different amount of MU to the same group of investors who face relatively less MU (i.e., $\vartheta = 1$). It is clear that the blue line is relatively flat, while the red dashed lines are hump-shaped, which suggests that PU can help explain the relative risky-asset holdings in two groups of investors only if these two groups of investors face significantly different amount of MU.[30] For example, only the red thick dashed line can generate a ratio of stock holdings between investors with at least a college degree and investors not finishing high school (which is 21.4 as shown in Table 4), while the other two lines (which represent smaller differences in MU) cannot. Similarly, without enough amount of PU (that is, if p is close to either 0 or 1), the model has difficulty in generating a high ratio of risky-asset holdings. So, this exercise shows that the interaction between MU and PU is important in explaining the relative risky-asset holding in the data.

and constraints, and avoid investment opportunities for which they feel unqualified.

[30] Note that $p = 0$ and $p = 1$ refer to cases without parameter uncertainty, while a value of p between 0 and 1 represents a positive amount of PU. This is why the red lines are hump-shaped and peak in the middle.

4.4 Welfare Cost of Ignorance

Comparing with the full information case, the investor with incomplete information about income growth makes consumption and portfolio decisions deviating from the first-best path. In other words, in this model, having more precise information about income growth can improve the investor's welfare. The following proposition provides the result on the investor's lifetime welfare under full information and model uncertainty:

Proposition 3 *Under full information and model uncertainty, the value function is given by*

$$\widetilde{V}\left(w_t, y_t; Z_t\right) = -\frac{\beta\psi}{r}\exp\left\{-\frac{r}{\psi}\left[w_t + g(y_t) + \phi(Z_t) + \frac{(\beta - r)\psi}{r^2} + \frac{\pi^2}{2\widetilde{\gamma}r^2\sigma_e^2}\right]\right\}, \tag{35}$$

where $\phi(Z_t)$ is the certainty equivalent human wealth of regime switching. Denote $\phi_1 = \phi(H)$ and $\phi_2 = \phi(L)$, then (ϕ_1, ϕ_2) jointly solve

$$r\phi_1 = \frac{\delta}{r+\rho} - \frac{\lambda_1\psi}{r}\left(\exp\left[\frac{r}{\psi}(\phi_1 - \phi_2)\right] - 1\right), \tag{36}$$

$$r\phi_2 = -\frac{\lambda_2\psi}{r}\left(\exp\left[-\frac{r}{\psi}(\phi_1 - \phi_2)\right] - 1\right). \tag{37}$$

Proof. See Online Appendix for the derivation. ■

Since more precise information on income growth leads to higher welfare, at $t = 0$ the investor would prefer a completely observable economy to a partially observable economy with the same initial conditions:

$$\widetilde{V}(w_0, y_0; Z_0) \geq V(w_0, y_0, p_0),$$

where $V(w_0, y_0, p_0)$ and $\widetilde{V}\left(w_t, y_t; Z_t\right)$ are provided in (34) and (35), respectively. To convert the welfare loss due to incomplete information about income growth into an equivalent wealth measure, we define *the value of information,* Π, as the additional amount of wealth needed in order for an investor with partial information to have the same life-time utility level as that with complete information. That is,

$$V(w_0 + \Pi, y_0, p_0) = \widetilde{V}(w_0, y_0; Z_0). \tag{38}$$

Solving Equation (38) gives the following expression for Π:

$$\Pi = \phi(Z_0) - f(p_0). \tag{39}$$

Since Z_0 is a Markov chain, we use the mathematical expectations of Π with respect to Z_0 to measure the value of information in our model. Specifically, using the distribution of Z_0 in

the partial information setting, $P(Z_0 = H) = p_0$ and $P(Z_0 = L) = 1 - p_0$, the mathematical expectation of Π can be written as

$$\varepsilon = E_Z [\Pi] = [p_0\phi_1 + (1 - p_0)\phi_2] - f(p_0). \tag{40}$$

The upper panel of Figure 9 illustrates how ε varies with the initial belief for different degrees of intertemporal substitution (ψ) and robustness (ϑ). It is clear from this figure that ε is concave in initial belief p_0, i.e., when the agent is more uncertain about the current regime, the value of information is higher. When $p_0 = 0$ or $p_0 = 1$, ε is also positive, which means that information is valuable even if the agent has no uncertainty about the current regime. In addition, we can see from the figure that the information value is skewed and more left skewed with the larger degrees of intertemporal substitution and robustness. For given values of p_0, it is clear that ε increases with the EIS. The intuition behind this result is that the lower the value of EIS, the larger its impact on model uncertainty, and the less the value of $f(p_0)$. However, the values of ϕ_1 and ϕ_2 also increase with EIS.[31] The net effect of these two mechanisms is that ε increases with the EIS.

To evaluate the relative importance of incomplete information and robustness in determining the welfare loss due to ignorance, we need to decompose the value of information into two components: one is purely from incomplete information and the other is due to robustness. Specifically, denote ε_0 as the information value for the uncertainty-neutral agent and ε_L as the welfare loss due to robustness in the incomplete information case. We can then decompose ε as:

$$\varepsilon = \varepsilon_0 + \varepsilon_L, \tag{41}$$

where ε_0 is obtained from the information value ε by setting $\vartheta = 0$, and ε_L is the difference between ε and ε_0 and measures the welfare loss due to robustness. The lower panel of Figure 9 shows that the ratio of the information-value loss ε_L to the information value ε, $\varepsilon_L/\varepsilon$, is convex in initial belief p_0 for different values of ψ and ϑ. The more uncertain about the current regime the agent is, the larger the amount of parameter uncertainty is, and the lower the ratio $\varepsilon_L/\varepsilon$ is. The ratio $\varepsilon_L/\varepsilon$ appears to be convex in the initial belief, p_0; it is lower when the agent's belief is near 0.5 and higher when his belief is close to 0 or 1. Furthermore, the figure also shows that the ratio decreases with the EIS and increases with the degree of robustness. The reason for this result is that $\varepsilon_L = n(p_0) - f(p_0)$, where $n(p_0)$ is defined in (23), decreases with the degree of EIS because $n(p_0)$ is independent of the degrees of EIS and RB. The impact of EIS on ε_L dominates the impact of EIS on ε; consequently, the ratio decreases with the degree of EIS.

[31] We can verify that ϕ_1 and ϕ_2 are increasing functions of ψ by solving the equation system, (36) and (37), numerically. The detailed derivations are available from the corresponding author.

5 Unknown Gaussian Income Growth

5.1 Model Specification and Theoretical Implications

In this section, we consider an extension in which the unobservable income growth rate follows a continuous-state stochastic process. Specifically, we assume that the growth rate μ_t follows a mean-reverting Ornstein-Uhlenbeck process:

$$d\mu_t = \lambda\left(\overline{\mu} - \mu_t\right) dt + \sigma_\mu dB_{\mu,t}, \tag{42}$$

where $B_{\mu,t}$ is a standard Brownian motions defined over the complete probability space, λ and $\overline{\mu}$ are positive constants, and the correlation between $dB_{y,t}$ and $dB_{\mu,t}$ is $\rho_{y\mu}$. Hence, in this setting both the income growth rate and the actual income are stochastic and risky.

As in the benchmark model, here μ_t is also unknown to investors. In this case, the investors need to estimate it using their observations of the realized labor income. Specifically, the typical investor estimates the conditional distribution of the true income growth rate and then represents the investor's original optimizing problem as a Markovian one. If we assume that the loss function in our model is the mean square error (MSE) due to incomplete information, then given a Gaussian prior, finding the posterior distribution of the income growth rate becomes a standard Kalman-Bucy filtering problem. However, if we assume that model uncertainty due to robustness not only affects the optimal control problem but also affects the optimal filtering problem, we have to consider a robust filtering problem. Following Kasa (2006) and Hansen and Sargent (Chapter 17, 2007), we now consider a situation in which the investor pursues a robust Kalman gain. To obtain robust Kalman filter gain, the investor considers the following distorted model:

$$d\mu_t = \left[\lambda\left(\overline{\mu} - \mu_t\right) + \sigma_\mu v_{1,t}\right] dt + \sigma_\mu d\widetilde{B}_{\mu,t}, \tag{43}$$

$$dy_t = \left(\mu_t - \rho y_t + \sigma_y v_{2,t}\right) dt + \sigma_y d\widetilde{B}_{y,t}, \tag{44}$$

where $\widetilde{B}_{\mu,t}$ and $\widetilde{B}_{y,t}$ are Wiener processes that are related to the approximating processes as follows

$$\widetilde{B}_{\mu,t} = B_{\mu,t} - \int_0^t v_{1,s} ds \text{ and } \widetilde{B}_{y,t} = B_{y,t} - \int_0^t v_{2,s} ds$$

and $\nu_{1,t}$ and $\nu_{2,t}$ are distortions to the conditional means of the two shocks, $\widetilde{B}_{\mu,t}$ and $\widetilde{B}_{y,t}$, respectively. As shown in Basar and Bernhard (1995) and Kasa (2006), a robust filter can be characterized by the following dynamic zero-sum game:

$$J_t = \inf_{\{m_j\}} \sup_{\{v_{1,j}, v_{2,j}\}} \left\{ \limsup_{T\to\infty} \frac{1}{T} E^Q \int_0^T \left(\mu_j - m_j\right)^2 dj - \vartheta^{-1} H\left(Q|P\right), \right\}, \tag{45}$$

subject to (43) and (44), where Q and P denote the distorted and approximating models, respectively. The relative entropy constraint is defined as:

$$H\left(Q|P\right)=\limsup_{T\to\infty}\frac{1}{2T}E^{Q}\left[\int_{0}^{T}\left(v_{1,t}^{2}+v_{2,t}^{2}\right)dj\right]\leq\eta_{0}, \tag{46}$$

where η_0 defines the set of models that the consumer is considering, and ϑ^{-1} is the Lagrange multiplier on the relative entropy constraint, (46). As shown in Dai Pra, Meneghini, and Runggaldier (1996), the entropy constrained robust filtering problem, (45), is equivalent with the following risk-sensitive filtering problem:

$$\frac{1}{\vartheta}\log\left(\int\exp\left(\vartheta F\left(\mu,m\right)\right)dP\right)=\sup_{Q}\left\{\int F\left(\mu,m\right)dQ-\vartheta^{-1}H\left(Q|P\right)\right\}, \tag{47}$$

where $F\equiv\left(\mu-m\right)^{2}/\vartheta$ is the loss function. The following proposition summarizes the results for this robust filtering problem:

Proposition 4 *When $\vartheta\geq\sigma_{\mu}^{2}/(\sigma_{\mu}^{2}+\lambda^{2})$, there is a unique solution for the robust filtering problem, (47):*

$$dm_{t}=\lambda\left(\overline{\mu}-m_{t}\right)dt+K_{t}\left[dy_{t}-\left(m_{t}-\rho y_{t}\right)dt\right], \tag{48}$$

$$\frac{d\Sigma_{t}}{dt}=-2\lambda\Sigma_{t}-\left(\frac{1}{\sigma_{y}^{2}}-\vartheta\right)\left(\eta+\Sigma_{t}\right)^{2}+\sigma_{\mu}^{2}, \tag{49}$$

where $\Sigma_t=E_t\left[\left(\mu_t-m_t\right)^2\right]$ is the conditional variance of μ_t, $K_t=\left(\eta+\Sigma_t\right)/\sigma_y^2$, is the Kalman gain, and $\eta=\rho_{y\mu}\sigma_y\sigma_\mu$. In the steady state, the conditional variance converges to

$$\Sigma^{*}=-\eta+\frac{-\lambda\sigma_{y}^{2}+\sigma_{y}\sqrt{\lambda^{2}\sigma_{y}^{2}+\left(1-\vartheta\sigma_{y}^{2}\right)\left(2\lambda\eta+\sigma_{\mu}^{2}\right)}}{1-\vartheta\sigma_{y}^{2}}, \tag{50}$$

which is non-negative.

Proof. From (47), it is clear that the right-hand side is an entropy constrained filtering problem in terms of the scaled quadratic objective function F, while the left-hand side is a risk-sensitive filtering problem in terms of the same function F, with risk-sensitivity parameter, ϑ. The solution of this risk-sensitive filtering problem is just a special case of Theorem 3 of Pan and Basar (1996). ■

Using the expected mean m_t, we can now rewrite the income process as:

$$dy_{t}=\left(m_{t}-\rho y_{t}\right)dt+\sigma_{y}dB_{m,t}, \tag{51}$$

where $dB_{m,t} = dB_{y,t} + \left(\frac{\mu_t - m_t}{\sigma_y}\right)$ is the normalized unanticipated innovation of the income growth process and it is a standard Brownian motion with respect to the investor's filtration. Substituting it into the estimated state updating equation yields:

$$dm_t = \lambda\left(\overline{\mu} - m_t\right)dt + \sigma_m dB_{m,t}, \tag{52}$$

where

$$\sigma_m \equiv \frac{\eta + \Sigma^*}{\sigma_y} = \frac{-\lambda\sigma_y + \sqrt{\lambda^2\sigma_y^2 + \left(1 - \vartheta\sigma_y^2\right)\left(2\lambda\eta + \sigma_\mu^2\right)}}{1 - \vartheta\sigma_y^2} \tag{53}$$

is the diffusion coefficient and Σ^* is given in (50). From (50), it is straightforward to show that

$$\frac{\partial\sigma_m}{\partial\vartheta} > 0 \text{ and } \frac{\partial\left(\sigma_m/\sigma_y\right)}{\partial\vartheta} > 0.$$

That is, the robust Kalman gain, σ_m/σ_y, is increasing with the preference for pursuing robust Kalman filter. In the next subsection, we will quantitatively examine the relative importance of robust control and filtering in determining precautionary savings and strategic asset allocation after using the US data to estimate the joint y and μ process.

Following the standard procedure, we can then solve for robust consumption-portfolio rules for this unknown Gaussian income growth case. The following proposition summarizes the solution:

Proposition 5 *With unknown income growth, the robust consumption and portfolio rules are*

$$c_t^* = r\left(w_t + h_t + l_t\right) + \Psi + \Pi - \Gamma, \tag{54}$$

$$\alpha_t^* = \alpha_s + \alpha_y + \alpha_p, \tag{55}$$

where $h_t = \frac{1}{r+\rho}\left(y_t + \frac{\overline{\mu}}{r} - \frac{\pi\rho_{ye}\sigma_y}{r\sigma_e}\right)$ *is the risk-adjusted human wealth,* $l_t = \frac{1}{(r+\rho)(\lambda+r)}\left(m_t - \overline{\mu} - \frac{\pi\rho_{ye}\sigma_m}{r\sigma_e}\right)$, $\Psi = \left(\frac{\beta - r}{r}\right)\psi$, $\Pi = \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}$,

$$\Gamma \equiv \frac{1}{2}r\widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{(r+\rho)(\lambda+r)}\right]^2 \tag{56}$$

is the investor's precautionary saving demand, and $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$ *is the effective coefficient of absolute risk aversion,*

$$\alpha_s = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} \tag{57}$$

is the standard speculation demand for the risky asset,

$$\alpha_y = -\frac{\rho_{ye}\sigma_y}{\sigma_e\left(r+\rho\right)} \tag{58}$$

is the labor income-hedging demand,

$$\alpha_p \equiv -\frac{\rho_{ye}\sigma_m}{\sigma_e\left(r+\rho\right)\left(r+\lambda\right)} \tag{59}$$

is the learning-induced hedging demand, and $\sigma_m = -\lambda\sigma_y + \sqrt{\lambda^2\sigma_y^2 + 2\lambda\eta + \sigma_\mu^2}$.

Proof. See Online Appendix B for the derivation. ■

It is clear from (56) that the preference for robustness (ϑ) affects the precautionary saving demand via two channels: (i) the direct channel (the robust control channel) and (ii) the indirect channel (the robust filtering channel) via the robust Kalman gain (σ_m/σ_y).[32] In addition, both the correlation between the equity return and labor income (ρ_{ye}) and the correlation between labor income and the mean growth rate of labor income ($\rho_{y\mu}$) interact with fundamental uncertainty (σ_y) and parameter uncertainty (σ_m), and then affect the precautionary saving demand. To fully explore how parameter uncertainty due to unknown income growth affects the precautionary saving demand, we first shut down the parameter uncertainty channel. In this case, the investor has complete information about the parameter, and the precautionary saving demand, Γ_0, can be written as follows:[33]

$$\Gamma_0 = \frac{1}{2} r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho}+\frac{\rho_{y\mu}\sigma_\mu}{(r+\rho)(\lambda+r)}\right]^2. \tag{60}$$

We then define $\Gamma_p \equiv \Gamma - \Gamma_0$ the additional demand for precautionary saving due to the ignorance of the unknown parameter μ. Using (56) and (60), it is straightforward to show that

$$\Gamma_p = \frac{1}{2} r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\left[\frac{2\sigma_y}{r+\rho}+\frac{\sigma_m+\rho_{y\mu}\sigma_\mu}{(r+\rho)(\lambda+r)}\right]\frac{\sigma_m-\rho_{y\mu}\sigma_\mu}{(r+\rho)(\lambda+r)}. \tag{61}$$

It is clear from (56), (60), and (61) that the precautionary saving demand is always decreasing with ρ_{ye} because hedging with the risky asset reduces the investor's precautionary saving demand. Furthermore, we can also see from (60) and (61) that Γ_0 and Γ_p increases and decreases with $\rho_{y\mu}$ for given ρ_{ye}, respectively. In the next subsection, we will evaluate the relative importance of this additional demand due to parameter uncertainty after estimating the y_t and μ_t process.

Expressions (57) and from (58) are the same as that obtained in our benchmark model in which the unknown parameter takes discrete values. (59) clearly shows how the importance of parameter uncertainty (σ_m) affects the learning-induced hedging demand for the risky asset (α_p). It is worth noting that RB also affects asset allocation via two channels: the direct channel (the robust control part) and the indirect channel (the robust filtering part). Specifically, model uncertainty due to robustness ($\widetilde{\gamma}$) does not enter both the income hedging demand and the learning-induced hedging demand. Using (58) and (59), the relative importance of these two types of asset demand can be written as:

$$\Xi \equiv \frac{\alpha_p}{\alpha_y} = \frac{\sigma_m}{\sigma_y(r+\lambda)}, \tag{62}$$

which does not depend on the degree of RB. It is worth noting that this result is different from that obtained in our benchmark model with Markov switching in which model uncertainty affects

[32] In the next subsection, we will quantitatively show that the indirect channel only has minor impact on precautionary saving and asset allocation and is dominated by the direct channel.

[33] Note that here when $\rho_{y\mu}=0$, the precautionary saving demand is the same as that obtained in Model I and III of Wang (2009). When $\rho_{ye}=\rho_{y\mu}=0$, the model reduces to that obtained in the standard Caballero (1990) model.

the relative importance of these two types of asset demand because the preference for robustness affects the belief updating equation.

5.2 Quantitative Implications

In order to estimate the persistence coefficients (ρ and λ) and the correlation coefficient ($\rho_{y\mu}$) in household income process, we estimate a discrete-time state-space system using the US households' disposable income from $1947 - 2006$: [34]

$$y_{t+1} = \mu_t + \rho_y y_t + \varepsilon_{t+1},$$
$$\mu_{t+1} = \rho_\mu \mu_t + \epsilon_{t+1},$$

where ε_{t+1} and ϵ_{t+1} are the iid normal innovations to the y and μ processes, respectively. The estimated values of the persistence coefficients, ρ_y and ρ_μ, are 0.9058 and 0.9675, respectively. The standard deviations of ε_{t+1} and ϵ_{t+1} are 0.0157 and 0.0056, respectively.[35] Using the estimates, we can recover the corresponding parameter values used in our continuous-time model: $\rho = 0.0989$, $\lambda = 0.0330$, $\sigma_y = 0.0165$, $\sigma_\mu = 0.0057$, and $\rho_{y\mu} = 0.8655$.

Using these estimated parameter values, we can examine how RB affects the robust Kalman gain in the filtering problem. Figure 10 shows how RB affects the robust Kalman gain (σ_m/σ_y) for different values of $\rho_{y\mu}$. It clearly shows that RB only has minor impact on the Kalman gain in the filtering problem.[36] The reason behind this result is that RB affects the Kalman gain via the $\vartheta\sigma_y^2$ term in (53) and the value of this term is very small given the estimated value of σ_y. Furthermore, we can also explore how induced uncertainty due to ignorance affects the different components of the precautionary saving demand and asset holdings. Figure 11 clearly illustrates that both the total and learning-induced precautionary saving components, Γ and Γ_p, increase with ϑ for various values of ρ_{ye}, whereas the relative importance of the learning-induced component does not change significantly when ϑ changes. The reason is that the effects of RB on Γ and Γ_p are similar and thus only have minor impact on the ratio of Γ_p to Γ. Furthermore, we can see from the figure that the learning-induced component is decreasing with the correlation between the equity return and labor income. This pattern is similar to that obtained in our benchmark model when the unknown mean of income growth follows a discrete distribution.

Figure 12 illustrates how RB affects the total demand for the risky asset (α) and the learning-induced hedging demand (α_p). It is clear from the figure that the total demand for the risky

[34] We exclude the period during and after the Great Recession (2007-2009) because the volatility was significantly larger than other years. But including that period does not change our key results.

[35] Following the literature, we also normalize household income measures as ratios of the mean for that year.

[36] This result is consistent with that obtained in Luo and Young (2016). They also find that RB does not play a significant role in affecting the robust Kalman gain within the linear-quadratic-Gaussian permanent income framework.

asset is decreasing with ϑ. In addition, we can also see that α is increasing with the correlation between the equity return and labor income. More importantly, we can see that the learning-induced hedging demand plays an important role in determining the strategic asset allocation. For example, when $\vartheta = 4$, the share of the learning-induced hedging demand accounts for more than 40 percent in the total asset holdings. From (62), it is straightforward to show that the learning-induced hedging demand (α_p) is much more important than the labor income-hedging demand (α_y). For example, when $\rho_{ye} = 0.6$, $\Xi = 9.4$, which means α_p is more than 9 times greater than α_y.

5.3 General Equilibrium Implications

In this extended model, we can fully explore the general equilibrium implications of ignorance. Following Huggett (1993), Calvet (2001), and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, of total mass normalized to one, with each agent solving the optimal consumption and savings problem with parameter and model uncertainty proposed in the previous subsection. Similar to Calvet (2001), we also make the following assumption:[37]

Assumption 1 *Both the risk-free asset and the risky asset in our model are in zero net supply. The initial cross-sectional distribution of disposable labor income is a stationary distribution* $\Phi(\cdot)$.

By the law of large numbers in Sun (2006), provided that the spaces of agents and the probability space are constructed appropriately, aggregate income and the cross-sectional distribution of permanent income $\Phi(\cdot)$ are constant over time. To fully explore the general equilibrium implications in this two-asset case, we first consider the equilibrium in the market for the risky asset. Assuming that the net supply of the risky asset is 0, the equilibrium condition in the market for the risky asset can be written as:

$$\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_y}{\sigma_e(r+\rho)} - \frac{\rho_{ye}\sigma_m}{\sigma_e(r+\rho)(r+\lambda)} = 0 \tag{63}$$

for a given risk free rate, r. From this equilibrium condition, we can express the equilibrium risky premium (π^*) in terms of the risk free rate:

$$\pi^* = \frac{r}{r+\rho}\widetilde{\gamma}\rho_{ye}\sigma_e\sigma_y(1+x), \tag{64}$$

where $x = \frac{\sigma_m}{(\lambda+r)\sigma_y} > 0$.

We next discuss how the equilibrium risk free rate, r, is determined in the market for the risk free asset. Using the individual consumption function (54) and the budget constraint, we can

[37] Huggett (1993) and Wang (2003) do not model the risky asset in their heterogenous-agent model economies.

obtain the expression for the individual saving function as follows:

$$d_t^* = f_t(r) + f_{m,t}(r) - \Psi + \Gamma_0, \tag{65}$$

where $f_t(r) \equiv \frac{\rho(y_t - \overline{y})}{r+\rho}$, $f_{m,t}(r) \equiv -\frac{r(m_t - \overline{\mu})}{(r+\rho)(\lambda+r)}$,

$$\Psi \equiv \psi\left(\frac{\beta}{r} - 1\right) \tag{66}$$

captures the saving demand of relative patience,

$$\Gamma_0 \equiv \frac{1}{2} r\widetilde{\gamma}\left(\frac{\sigma_y}{r+\rho}\right)^2 (1+x)^2 \tag{67}$$

is the precautionary saving demand when $\rho_{ye} = 0$. Following the aggregation procedure used in Huggett (1993) and Wang (2003), we have the following result on the total saving demand:

Proposition 6 *Both the total demand of savings "for a rainy day" and the total demand for the estimation-risk-induced savings equal zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} f_t(r)\, d\Phi(y_t) = 0$ *and* $F_{m,t}(r) = \int_{m_t} f_{m,t}(r)\, d\Phi(m_t) = 0$, *for* $r > 0$.

Proof. The proof uses the LLN and is the same as that in Wang (2003). ■

Using this result, from (65), after aggregating across all investors, the expression for total savings can be written as

$$D(\vartheta, r) \equiv \Gamma_0(\vartheta, r) - \Psi(r), \tag{68}$$

where $\Gamma_0(\vartheta, r)$ and $\Pi(\vartheta, r)$ are given in (67) and (66), respectively. An equilibrium interest rate r^* satisfies $D(\vartheta, r^*) = 0$. The following proposition proves that an equilibrium exists:

Proposition 7 *There exists one equilibrium with an interest rate* $r^* \in (0, \beta)$ *such that* $D(\vartheta, r^*) = 0$, *and*

$$\pi^* = \frac{r^*}{r^*+\rho}\widetilde{\gamma}\rho_{ye}\sigma_e\left(\sigma_y + \frac{\sigma_m}{r^*+\lambda}\right), \tag{69}$$

In any such equilibrium, each consumer's optimal consumption-portfolio rules are described by:

$$c_t^* = r^*(w_t + h_t + l_t), \tag{70}$$

and

$$\alpha^* = 0, \tag{71}$$

respectively, where

$$h_t^* = \frac{1}{r^*+\rho}\left(y_t + \frac{\overline{\mu}}{r^*}\right), \tag{72}$$

$$l_t^* = \frac{1}{(r^*+\rho)(\lambda+r^*)}(m_t - \overline{\mu}), \tag{73}$$

and $\widetilde{\gamma} \equiv (1+\vartheta)\gamma$ *is the effective coefficient of absolute risk aversion.*

Proof. If $r > \beta$, $\Gamma_0(\vartheta, r)$ and $-\Psi(r)$ in the expression for total savings $D(\vartheta, r^*)$ are positive, which contradicts the equilibrium condition: $D(\vartheta, r^*) = 0$. Since $\Gamma_0(\vartheta, r) - \Psi(r) < 0$ (> 0) when $r = 0$ $(r = \beta)$, the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \beta)$ such that $D(\vartheta, r^*) = 0$. To prove this equilibrium is unique, note that

$$\frac{\partial D(\vartheta, r)}{\partial r} = \frac{\psi}{r^2}\left[(\beta - r)\left(1 - \frac{2r}{r+\rho} - \frac{2r}{r+\lambda}\frac{x}{1+x}\right) + \beta\right]$$

where we use the equilibrium condition: $0.5r\widetilde{\gamma}(1+x)^2\sigma_y^2/(r+\rho)^2 = \psi(\beta - r)/r$, is positive in general equilibrium when

$$(\beta - r)\left(1 - \frac{2r}{r+\rho} - \frac{2r}{r+\lambda}\frac{x}{1+x}\right) + \beta > 0 \text{ or } \beta > \frac{r}{2}\left(\frac{\frac{\rho - r}{r+\rho} - \frac{2r}{r+\lambda}\frac{x}{1+x}}{\frac{\rho}{r+\rho} - \frac{r}{r+\lambda}\frac{x}{1+x}}\right),$$

which holds for plausibly estimated parameter values. Note that when $\sigma_m = 0$,

$$\frac{\partial D(\vartheta, r)}{\partial r} > 0$$

always holds. Therefore, if $r < \beta$, there is only one equilibrium in $(0, \beta)$. From Expression (24), we can obtain the individual's optimal consumption rule in general equilibrium as $c_t^* = r^*(w_t + h_t^* + l_t^*)$. ■

From the equilibrium condition,

$$\frac{1}{2}r^*\left(\gamma + \frac{\vartheta}{\psi}\right)\frac{\sigma_y^2}{(r^*+\rho)^2}\left[1 + \frac{\sigma_m}{(\lambda + r^*)\sigma_y}\right]^2 - \psi\left(\frac{\beta}{r^*} - 1\right) = 0, \tag{74}$$

it is straightforward to show that

$$\frac{dr^*}{d\vartheta} = -\frac{\beta - r^*}{r^*\widetilde{\gamma}}\left\{\frac{\psi}{r^{*2}}\left[(\beta - r^*)\left(1 - \frac{2r^*}{r^*+\rho} - \frac{2r^*}{r^*+\lambda}\frac{x}{1+x}\right) + \beta\right]\right\}^{-1} < 0 \tag{75}$$

for plausibly estimated parameter values. It is clear from this expression that r^* is decreasing in the degree of RB, ϑ. Furthermore, it is clear from (64) that RB can affect the risk premium via two distinct channels. The first channel is direct: It can increase the equity premium by increasing the effective coefficient of risk aversion, $\widetilde{\gamma}$. The second channel is indirect: It affects the risk premium via the general equilibrium risk-free rate, r^*. Holding r^* fixed, it is clear from that the equity premium is a linear function of $\widetilde{\gamma}$. The upper panel of Figure 13 shows that RB reduces the equilibrium interest rate and increases the equilibrium equity premium, respectively, for the estimated and calibrated parameter values reported in the last subsection.[38] In addition, we can also see that the EIS can amplify the effects of RB on the equilibrium asset returns. Given

[38] Here we assume that $\rho_{ye} = 0.4$, $\psi = 0.2$, and $\gamma = 3$. The main results do not change for different values of these parameters.

the plausibly estimated parameter values, we find that the direct channel dominate the indirect channel. Furthermore, the lower panel of Figure 13 illustrates how the degree of parameter uncertainty measured by σ_m/σ_y affects the equilibrium asset returns. It is clear from Figure 13 that both model uncertainty and parameter uncertainty have significant effects on the equilibrium asset returns. For example, when $\gamma = 3$ and $\psi = 0.2$, the risk-free rate reduces from 2.13% to 1.63%, and the equity premium increases from 2.76% to 4.26% when ϑ increases from 2 to 4.[39] For the case when $\gamma = 3$, $\psi = 0.2$, and $\vartheta = 2$, the risk-free rate reduces from 2.56% to 1.70%, and the equity premium increases from 2.32% to 3.14% when σ_m/σ_y increases from 0.25 to 0.45. It is worth noting that here we assume zero net supply of the risky aset only for theoretical and illustrative purposes. Allowing for a positive supply of the risky asset can drive up the risk premium and drive down the risk free rate in general equilibrium. (Using (64) and (68), it is straightforward to obtain these results.)

6 Conclusion

In this paper we have studied how the interaction of two types of ignorance-induced uncertainty affects strategic consumption-portfolio rules and precautionary savings in a continuous-time recursive utility model with uninsurable labor income. Specifically, we have explicitly solved the model to explore how the two types of ignorance-induced uncertainty interact with intertemporal substitution, risk aversion, and the correlation between the risky asset and labor income. We show they have distinct impacts on strategic asset allocation and precautionary savings as well as the equilibrium asset returns. Furthermore, for plausibly estimated and calibrated model parameters, we find that the welfare cost of ignorance for ordinary investors can be very large.

[39] Note that in the FI-RE case, $r^* = 3.11$ percent and $\pi^* = 0.272$.

7 Appendix

7.1 Solving the FI-RE Model with Recursive Utility and Unknown Income Growth

Guess that $J_t = -\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_1 f(p_t)$, the J function at time $t+\Delta t$ can thus be written as:[40]

$$\begin{aligned}
J_{t+\Delta t} &= J(w_{t+\Delta t}, y_{t+\Delta t}, p_{t+\Delta t}) = -\alpha_0 - \alpha_1 w_{t+\Delta t} - \alpha_2 y_{t+\Delta t} - \alpha_1 f(p_{t+\Delta t}) \\
&\approx -\alpha_0 - [\alpha_1 w_t + \alpha_1 (r w_t + y_t - c_t + \alpha_t \pi)\Delta t + \alpha_1 \sigma_e \alpha_t \Delta B_{e,t}] \\
&- \left[\alpha_2 y_t + \alpha_2 (\mu_2 + \delta p_t - \rho y_t)\Delta t + \alpha_2 \rho_{ye}\sigma_y \Delta B_{e,t} + \alpha_2 \sqrt{1-\rho_{ye}^2}\sigma_y \Delta \hat{B}_{i,t}\right] \\
&- \left[\begin{array}{c} \alpha_1 f(p_t) + \alpha_1 f'(p_t)\left([\lambda_2 - (\lambda_1+\lambda_2)p_t]\Delta t + \sigma_y^{-1}\delta p_t (1-p_t)\left(\rho_{ye}\Delta B_{e,t} + \sqrt{1-\rho_{ye}^2}\Delta \hat{B}_{i,t}\right)\right) \\ +\frac{1}{2}\alpha_1 f''(p_t)\left(\sigma_y^{-1}\delta p_t (1-p_t)\right)^2 \Delta t \end{array}\right].
\end{aligned}$$

Using the above expression for $J_{t+\Delta t}$ and assume that the time interval Δt goes to infinitesimal dt, we can compute the certainty equivalent of J_{t+dt} as follows:

$$\begin{aligned}
&\exp(-\gamma CE_t) = E_t[\exp(-\gamma J(s_{t+dt}))] \\
&= \exp\left(-\gamma E_t[-\alpha_1 w_{t+dt} - \alpha_2 y_{t+dt} - \alpha_1 f(p_{t+dt})] + \frac{1}{2}\gamma^2 \operatorname{var}_t[-\alpha_1 w_{t+dt} - \alpha_2 y_{t+dt} - \alpha_1 f(p_{t+dt})] + \gamma\alpha_0\right) \\
&= \exp\left(\gamma\alpha_0 - \gamma(\partial J)^T \cdot (s_t + E[ds_t]) + \frac{\gamma^2}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right] dt\right) \\
&= \exp(-\gamma J_t)\exp\left(-\gamma(\partial J)^T \cdot E[ds_t] + \frac{\gamma^2}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right] dt\right),
\end{aligned}$$

where $s_t = \begin{bmatrix} w_t & y_t & f(p_t) \end{bmatrix}^T$, $ds_t = \begin{bmatrix} dw_t & dy_t & df(p_t) \end{bmatrix}^T$, $\partial J = \begin{bmatrix} J_w & J_y & J_f \end{bmatrix}^T$, and Σ is defined by (12). The above equation implies that

$$CE_t[J_{t+dt}] = J_t + \left((\partial J)^T \cdot E[ds_t] - \frac{\gamma}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right] dt\right), \tag{76}$$

where we use the fact that

$$\begin{aligned}
f(p_{t+dt}) &= f(p_t) + f'(p_t)\,dp_t + \frac{1}{2}f''(p_t)(dp_t)^2 \\
&= f(p_t) + f'(p_t)\left([\lambda_2 - (\lambda_1+\lambda_2)p_t]\,dt + \sigma_y^{-1}\delta p_t(1-p_t)\left(\rho_{ye}dB_{e,t} + \sqrt{1-\rho_{ye}^2}d\hat{B}_{i,t}\right)\right) \\
&+ \frac{1}{2}f''(p_t)\left(\sigma_y^{-1}\delta p_t(1-p_t)\right)^2 dt.
\end{aligned}$$

Substituting the expression of CE_t into the HJB yields:

$$\beta V(J_t) = \sup_{\{c_t,\alpha_t\}}\{\beta V(c_t) + \mathcal{D}V(J_t)\}, \tag{77}$$

[40] Here $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal distributed variable.

where

$$\mathcal{D}V(J_t) = V'(J_t)\left((\partial J)^T \cdot E[ds_t] - \frac{\gamma}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right]\right)$$
$$= V'(J_t)\left(\begin{array}{c} \left[\begin{array}{c} -\alpha_1(rw_t + y_t - c_t + \alpha_t\pi) - \alpha_2(\mu_2 + \delta p_t - \rho y_t) \\ -\alpha_1[\lambda_2 - (\lambda_1+\lambda_2)p_t]f'(p_t) - \frac{1}{2}\alpha_1 f''(p_t)\left(\sigma_y^{-1}\delta p_t(1-p_t)\right)^2 \end{array}\right] \\ -\frac{\gamma}{2}\left[\begin{array}{c} \alpha_1^2\alpha_t^2\sigma_e^2 + \alpha_2^2\sigma_y^2 + \alpha_1^2\sigma_y^{-2}[f'(p_t)\delta p_t(1-p_t)]^2 \\ +2\alpha_1\sigma_e\alpha_t\alpha_2\rho_{ye}\sigma_y + 2\alpha_1\sigma_e\alpha_t\alpha_1 f'(p_t)\sigma_y^{-1}\delta p_t(1-p_t)\rho_{ye} \\ +2\alpha_1\alpha_2 f'(p_t)\delta p_t(1-p_t) \end{array}\right] \end{array}\right). \quad (78)$$

The FOC for c_t is then

$$c_t = -\psi\ln\left(-\frac{\alpha_1}{\beta}\right) + (-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_1 f(p_t)), \quad (79)$$

where we use the facts that $V(c_t) = (-\psi)\exp(-c_t/\psi)$ and $V(J_t) = (-\psi)\exp(-J_t/\psi)$. The FOC for α_t is

$$\alpha_t = -\frac{\pi}{\gamma\alpha_1\sigma_e^2} - \frac{\alpha_2\rho_{ye}\sigma_y}{\alpha_1\sigma_e} - \frac{\rho_{ye}\delta p_t(1-p_t)f'(p_t)}{\sigma_e\sigma_y}. \quad (80)$$

Substituting these FOCs back into the HJB and matching the w_t, y_t, and constant terms on both sides of the above equation yields

$$\alpha_1 = -r,\ \alpha_2 = -\frac{r}{r+\rho},\ \text{and}\ \alpha_0 = \left(1-\frac{\beta}{r}\right)\psi - \psi\ln\left(\frac{r}{\beta}\right) - \frac{\mu_2}{r+\rho} + \frac{\pi\rho_{ye}\sigma_y}{\sigma_e(r+\rho)} - \frac{\pi^2}{2r\gamma\sigma_e^2} + \frac{r\gamma}{2}\left(1-\rho_{ye}^2\right)\sigma_s^2.$$

Substituting these coefficients back to the FOCs, (79) and (80), yields the following optimal consumption and portfolio rules under FI-RE:

$$c_t = r\left[w_t + \frac{1}{r+\rho}\left(y_t + \frac{\mu_2}{r} - \frac{\pi\rho_{ye}\sigma_y}{r\sigma_e}\right) + rf(p_t)\right] + \Psi + \Pi - \Gamma,$$
$$\alpha_t = -\frac{\pi}{\gamma\alpha_1\sigma_e^2} - \frac{\alpha_2\rho_{ye}\sigma_y}{\alpha_1\sigma_e} - \frac{\rho_{ye}\delta p_t(1-p_t)f'(p_t)}{\sigma_e\sigma_y},$$

where $\Psi = \left(\frac{\beta}{r} - 1\right)\psi$, $\Pi = \frac{\pi^2}{2r\gamma\sigma_e^2}$, and $\Gamma = \frac{r\gamma}{2}\left(1-\rho_{ye}^2\right)\sigma_s^2$. Putting the terms including p_t together, we have

$$rf(p_t) = \frac{\delta}{r+\rho}p_t - \left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\gamma}{r+\rho}\left(1-\rho_{ye}^2\right)\right]f'(p_t)\delta p_t(1-p_t) + [\lambda_2 - (\lambda_1+\lambda_2)p_t]f'(p_t)$$
$$- \frac{r\gamma}{2\sigma_y^2}\left(1-\rho_{ye}^2\right)\left[f'(p_t)\delta p_t(1-p_t)\right]^2 + \frac{1}{2\sigma_y^2}f''(p_t)\left(\delta p_t(1-p_t)\right)^2,$$

which is just (13) in the main text.

7.2 Solving the Human Wealth under Incomplete Information

Under the risk-neutral probability measure $\mathbb{Q}$, we rewrite the dynamics of labor income (6) and belief updating process (7) as follows

$$dy_t = \left(\overline{\mu} - \rho y_t - \frac{\rho_{ye}\sigma_y \pi}{\sigma_e}\right) dt + \rho_{ye}\sigma_y dB^{\mathbb{Q}}_{e,t} + \sqrt{1-\rho_{ye}^2}\sigma_y d\hat{B}^{\mathbb{Q}}_{i,t}, \tag{81}$$

$$dp_t = \left[\lambda_2 - (\lambda_1+\lambda_2)\, p_t - \frac{\rho_{ye}\pi}{\sigma_y\sigma_e}\delta p_t\,(1-p_t)\right] dt + \sigma_y^{-1}\delta p_t\,(1-p_t)\left(\rho_{ye} dB^{\mathbb{Q}}_{e,t} + \sqrt{1-\rho_{ye}^2} d\hat{B}^{\mathbb{Q}}_{i,t}\right), \tag{82}$$

where $B^{\mathbb{Q}}_{e,t}$ and $\hat{B}^{\mathbb{Q}}_{i,t}$ are standard Brownian motions and mutually independent under the risk-neutral probability measure $\mathbb{Q}$ satisfying

$$dB^{\mathbb{Q}}_{e,t} = dB_{e,t} + \frac{\pi}{\sigma_e}dt \text{ and } d\hat{B}^{\mathbb{Q}}_{i,t} = d\hat{B}_{i,t}.$$

From the definition (20) of human wealth, $h(y,p)$ satisfies the following equation

$$\begin{aligned} rh(y,p) = \;& y + \left(\overline{\mu} - \rho y - \tfrac{\rho_{ye}\sigma_y\pi}{\sigma_e}\right) h_y + \left[\lambda_2 - (\lambda_1+\lambda_2)\, p - \tfrac{\rho_{ye}\pi}{\sigma_y\sigma_e}\delta p\,(1-p)\right] h_p \\ & + \tfrac{1}{2}\sigma_y^2 h_{yy} + \delta p(1-p)h_{yp} + \tfrac{1}{2\sigma_y^2}\delta^2 p^2(1-p)^2 h_{pp}, \end{aligned} \tag{83}$$

where h_y, h_p, h_{yy}, h_{pp}, and h_{yp} are the first and second partial derivatives of $h(y,p)$ with respect to y and p. Conjecture $h(y,p)$ is additive in income y and belief p, in that

$$h(y,p) = m(y) + n(p).$$

Conjecture $m(y)$ is affine in labor income y, we can derive the expression of $m(y)$ in (22). Substituting (22) into (83) we can obtain the differential equation (23) for $n(p)$. Substituting $p=0$ and $p=1$ into (23) gives the boundary conditions in Proposition 2.

Similar to Appendix 7.3, when δ is small, the approximation solution of $n\,(p_t)$ is

$$n\,(p_t) \approx n_1\,(p_t)\,\delta + n_2\,(p_t)\,\delta^2. \tag{84}$$

where

$$n_1\,(p_t) = \frac{1}{(r+\rho)\,(r+\lambda_1+\lambda_2)}\left(p_t + \frac{\lambda_2}{r}\right), \quad n_2\,(p_t) = b_0 + b_1 p_t + \frac{1}{2}b_2 p_t^2.$$

and

$$b_0 = \frac{\lambda_2}{r}b_1,\; b_1 = -\frac{b_2}{2}\frac{r+2\lambda_1}{r+\lambda_1+\lambda_2},\; b_2 = \frac{\rho_{ye}\pi}{\sigma_e\sigma_y}\frac{2}{(r+\rho)\,(r+\lambda_1+\lambda_2)\,(r+2\,(\lambda_1+\lambda_2))}.$$

When $\lambda_1 = \lambda_2 = 0$ (the constant and unknown income growth case), we have

$$n(p_t) \approx \frac{1}{r(r+\rho)}p_t\delta - \frac{\rho_{ye}\pi}{\sigma_e\sigma_y}\frac{1}{(r+\rho)r^2}\left(p_t - p_t^2\right)\delta^2.$$

7.3 Solving the RB-RU Model with Unknown Income Growth

Under RB, the HJB can be written as:

$$\beta V\left(J_t\right)=\sup_{\{c_t,\alpha_t\}}\inf_{\upsilon_t}\left\{\beta V\left(c_t\right)+\mathcal{D}V\left(J_t\right)+\frac{1}{2\vartheta_t}\left(\upsilon_t^T\cdot\Sigma\cdot\upsilon_t\right)\right\}, \tag{85}$$

where

$$\mathcal{D}V\left(J_t\right)=V'\left(J_t\right)\left(\left(\partial J\right)^T\cdot E\left[ds_t\right]+\upsilon_t^T\cdot\Sigma\cdot\partial J-\frac{\gamma}{2}\left[\left(\partial J\right)^T\cdot\Sigma\cdot\partial J\right]\right).$$

where $\upsilon_t=\begin{bmatrix}\upsilon_{1,t} & \upsilon_{2,t} & \upsilon_{3,t}\end{bmatrix}^T$, $\upsilon_t^T\cdot\Sigma\cdot\partial J$ is the adjustment to the expected continuation value when the state dynamics is governed by the distorted model with the mean distortion υ_t, and the final term, $\left(\upsilon_t^T\cdot\Sigma\cdot\upsilon_t\right)/2\vartheta_t$, quantifies the penalty due to RB.

Solving first for the infimization part of the robust HJB equation (85) yields:

$$\upsilon_t=-\vartheta_t V'\left(J_t\right)\partial J \tag{86}$$

Substituting this optimal distortion into (85) yields:

$$\beta V\left(J_t\right)=\sup_{\{c_t,\alpha_t\}}\left\{\beta V\left(c_t\right)+V'\left(J_t\right)\left(\partial J\cdot E\left[ds_t\right]-\frac{\gamma}{2}\left[\partial J\cdot\Sigma\cdot\left(\partial J\right)^T\right]-\frac{\vartheta_t}{2}V'\left(J_t\right)\left[\left(\partial J\right)^T\cdot\Sigma\cdot\partial J\right]\right)\right\}.$$

Following Uppal and Wang (2003) and Maenhout (2004), we adopt the normalization method and assume that

$$\vartheta_t=-\frac{\vartheta}{V\left(J_t\right)}.$$

Given that $V\left(J_t\right)=\left(-\psi\right)\exp\left(-J_t/\psi\right)$ and $V'\left(J_t\right)=\exp\left(-J_t/\psi\right)$, the HJB equation reduces to

$$\beta V\left(J_t\right)=\sup_{\{c_t,\alpha_t\}}\left\{\beta V\left(c_t\right)+V'\left(J_t\right)\left(\partial J\cdot E\left[ds_t\right]-\frac{1}{2}\widetilde{\gamma}\left[\partial J\cdot\Sigma\cdot\left(\partial J\right)^T\right]\right)\right\},$$

where $\widetilde{\gamma}=\gamma+\vartheta/\psi$.

The FOC for c_t is then

$$c_t=-\psi\ln\left(-\frac{\alpha_1}{\beta}\right)+\left(-\alpha_0-\alpha_1 w_t-\alpha_2 y_t-\alpha_1 f\left(p_t\right)\right), \tag{87}$$

where we use the facts that $V\left(c_t\right)=\left(-\psi\right)\exp\left(-c_t/\psi\right)$ and $V\left(J_t\right)=\left(-\psi\right)\exp\left(-J_t/\psi\right)$. The FOC for α_t is

$$\alpha_t=-\frac{\pi}{\widetilde{\gamma}\alpha_1\sigma_e^2}-\frac{\alpha_2\rho_{ye}\sigma_y}{\alpha_1\sigma_e}-\frac{\rho_{ye}\delta p_t\left(1-p_t\right)f'\left(p_t\right)}{\sigma_e\sigma_y}. \tag{88}$$

Substituting these FOCs back into the HJB and matching the w_t, y_t, and constant terms on both sides of the above equation yields

$$\alpha_1=-r,\ \alpha_2=-\frac{r}{r+\rho},\text{ and }\alpha_0=\left(1-\frac{\beta}{r}\right)\psi-\psi\ln\left(\frac{r}{\beta}\right)-\frac{\mu_2}{r+\rho}+\frac{\rho_{ye}\sigma_s\pi}{\sigma_e}-\frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}+\frac{r\widetilde{\gamma}}{2}\left(1-\rho_{ye}^2\right)\sigma_s^2.$$

Substituting these coefficients back to the FOCs, (79) and (80), yields the optimal consumption and portfolio rules under FI-RE. Putting the terms including p_t together, we have

$$rf(p_t) = \frac{\delta}{r+\rho}p_t - \left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}}{r+\rho}\left(1-\rho_{ye}^2\right)\right] f'(p_t)\,\delta p_t(1-p_t) + [\lambda_2 - (\lambda_1+\lambda_2)p_t]\, f'(p_t)$$
$$- \frac{r\widetilde{\gamma}}{2\sigma_y^2}\left(1-\rho_{ye}^2\right)\left[f'(p_t)\,\delta p_t(1-p_t)\right]^2 + \frac{1}{2\sigma_y^2}f''(p_t)\left(\delta p_t(1-p_t)\right)^2, \tag{89}$$

which is just (13) in the main text.

Following Wang (2009), when δ is small, we can expand $f(p_t)$ in terms of the power series of δ:

$$f(p_t) \approx f_1(p_t)\,\delta + f_2(p_t)\,\delta^2. \tag{90}$$

Plugging this approximation into (89) and keeping the terms up to δ^2 yield:

$$r\left(f_1(p_t)\,\delta + f_2(p_t)\,\delta^2\right) = \frac{\delta}{r+\rho}p_t - \left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{r+\rho}\right]\delta^2 p_t(1-p_t)\,f_1'(p_t) \tag{91}$$
$$+ [\lambda_2 - (\lambda_1+\lambda_2)p_t]\left(f_1'(p_t)\,\delta + f_2'(p_t)\,\delta^2\right)$$

Matching the coefficients for the δ and δ^2 terms yields:

$$rf_1(p_t) = \frac{1}{r+\rho}p_t + [\lambda_2 - (\lambda_1+\lambda_2)p_t]\, f_1'(p_t),$$
$$rf_2(p_t) = -\left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{r+\rho}\right]p_t(1-p_t)\,f_1'(p_t) + [\lambda_2 - (\lambda_1+\lambda_2)p_t]\, f_2'(p_t),$$

respectively. It is straightforward to show that

$$f_1(p_t) = \frac{1}{(r+\rho)(r+\lambda_1+\lambda_2)}\left(p_t + \frac{\lambda_2}{r}\right).$$

We now conjecture that $f_2(p)$ takes the following quadratic form: $f_2(p) = b_0 + b_1 p_t + \frac{1}{2}b_2 p_t^2$. Plugging this expression into the ODE for $f_2(p_t)$ yields:

$$r\left(b_0 + b_1 p_t + \frac{1}{2}b_2 p_t^2\right) = -\left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{r+\rho}\right]\frac{p_t(1-p_t)}{(r+\rho)(r+\lambda_1+\lambda_2)}$$
$$+ [\lambda_2 - (\lambda_1+\lambda_2)p_t](b_1 + b_2 p_t).$$

Matching the constant, p_t, and p_t^2 terms yields:

$$b_0 = \frac{\lambda_2}{r}b_1,$$
$$b_1 = -\frac{b_2}{2}\frac{r+2\lambda_1}{r+\lambda_1+\lambda_2},$$
$$b_2 = \left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{r+\rho}\right]\frac{2}{(r+\rho)(r+\lambda_1+\lambda_2)(r+2(\lambda_1+\lambda_2))}.$$

Note that when $\lambda_1 = \lambda_2 = 0$ (i.e., the constant and unknown income growth case), we have

$$b_0 = 0, b_1 = -\frac{b_2}{2}, \text{ and } b_2 = \left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{r+\rho}\right]\frac{2}{(r+\rho)r^2},$$

which means that $f_2\left(p_t\right) = -\frac{b_2}{2}\left(p_t - p_t^2\right) = -\left(p_t - p_t^2\right)\left[\frac{\rho_{ye}\pi}{\sigma_e\sigma_y} + \frac{r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{r+\rho}\right]\frac{1}{(r+\rho)r^2}$.

Using the expressions for $n\left(p_t\right)$ and $f\left(p_t\right)$, we can obtain that

$$\begin{aligned}\Gamma_p &\equiv r\left[n(p_t) - f\left(p_t\right)\right] = r\left[\left(n_1\left(p_t\right)\delta + n_2\left(p_t\right)\delta^2\right) - \left(f_1\left(p_t\right)\delta + f_2\left(p_t\right)\delta^2\right)\right] \\ &= -\frac{r^2\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{(r+\rho)^2\left(r+\lambda_1+\lambda_2\right)\left(r+2\left(\lambda_1+\lambda_2\right)\right)}\left(p_t^2 - \frac{r+2\lambda_1}{r+\lambda_1+\lambda_2}p_t - \frac{r+2\lambda_1}{r+\lambda_1+\lambda_2}\frac{\lambda_2}{r}\right),\end{aligned}$$

which is just (29) in the main text. Note that this expression reduces to $\Gamma_p = -\frac{\widetilde{\gamma}\left(1-\rho_{ye}^2\right)}{(r+\rho)^2}\left(p_t^2 - p_t\right)$ when $\lambda_1 = \lambda_2 = 0$.

7.4 Data Description

To construct our sample from the PSID, we include data only from the years in which the PSID has wealth information available: 1983, 1993, and biennially from $2000 - 2010$. We also exclude any households in the PSID poverty or Latino subsamples. Additional excluded households include female headed households and households experiencing a change in the head or family composition. If households are missing information on education, region, or income, they are also excluded. Age outliers are excluded by removing households with a husband or wife less than 30 or over 65. A household is also removed if they report a negative value for income. We also exclude some outliers in the sample. Our final sample contains $5,938$ unique households with $18,481$ observations.

In the PSID, wealth is defined as the sum of six asset types, net of debt value, plus the value of home equity for a given household. Home equity is calculated by subtracting the mortgage from the value of the home. The six asset types included are the value of a household's farm/business, the sum of all checking/savings accounts of all household members, the value of real estate owned by the household (besides their primary home), the value of any stocks owned by household members (which includes stock in publicly held corporations, mutual funds, or investment trusts such as IRAs), the value of all vehicles owned by the household, and any other assets the household owns. We also examine a measure of wealth that is constructed with the same method, but excludes home equity.

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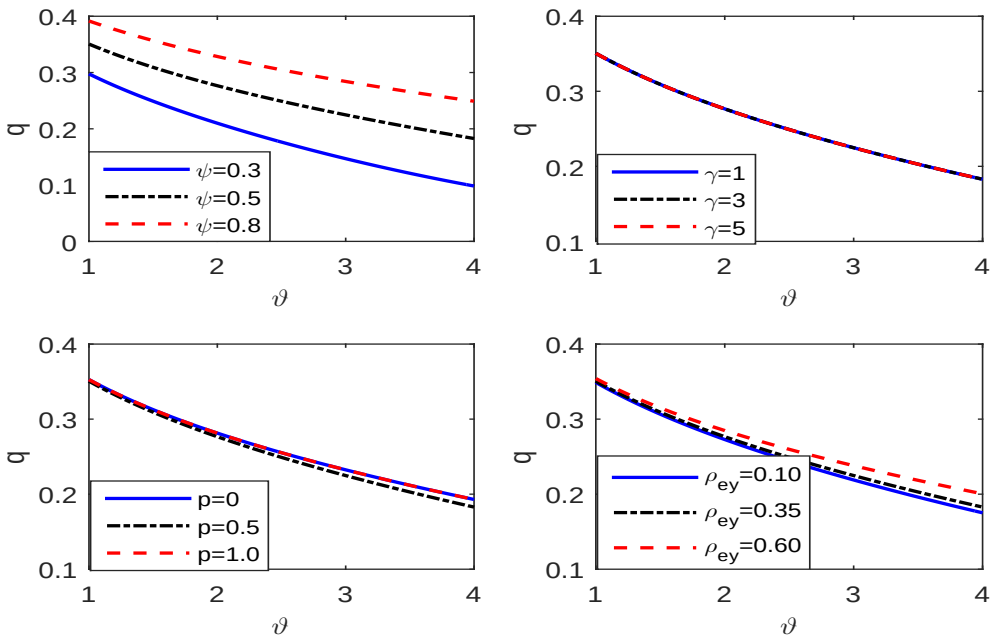

Figure 1: Relationship between ϑ and q

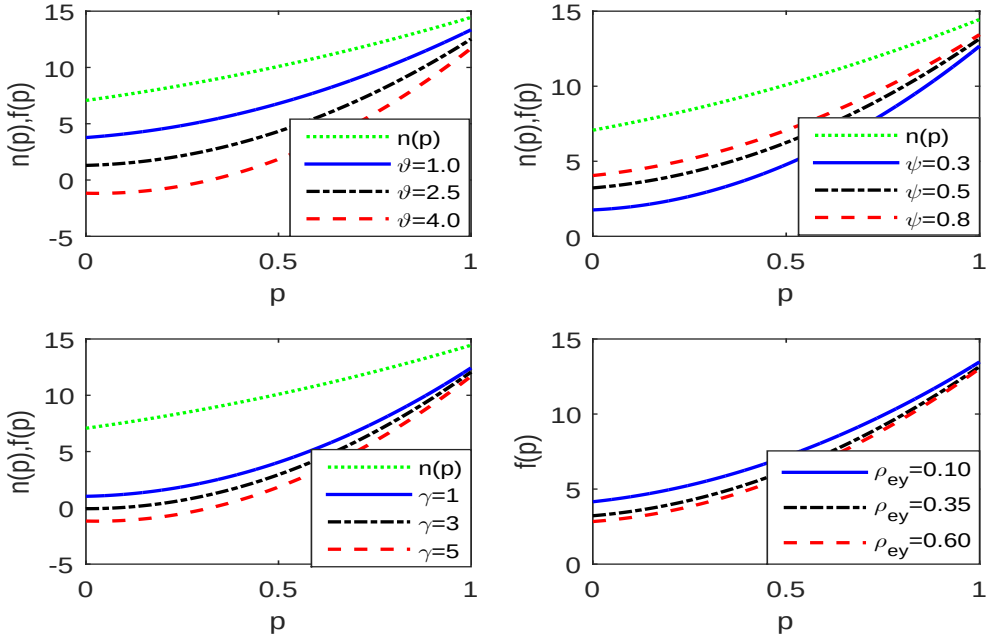

Figure 2: Effects of RB on $f(p)$

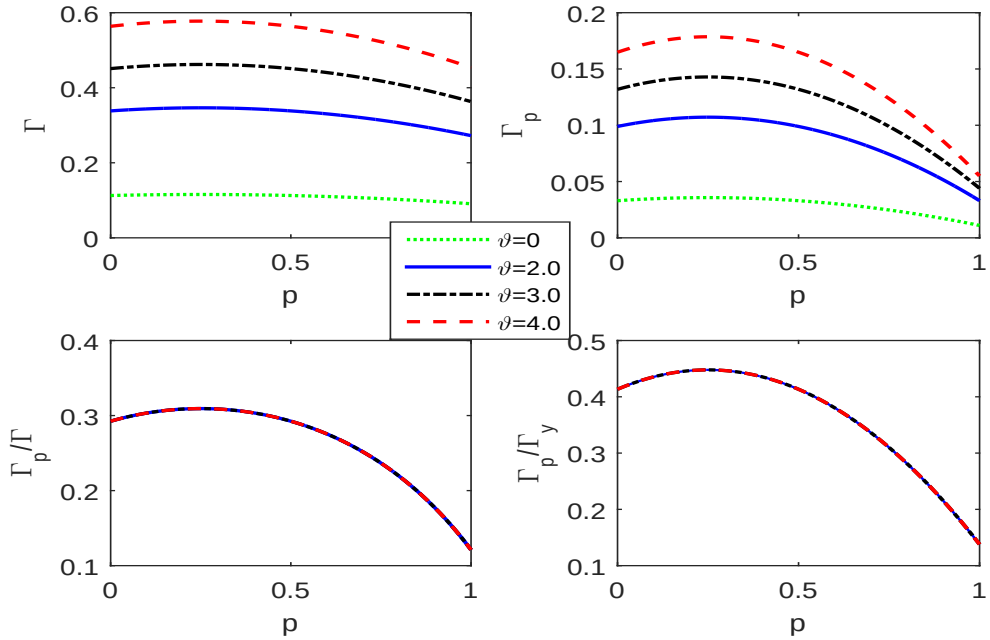

Figure 3: Effects of RB on Precautionary Saving

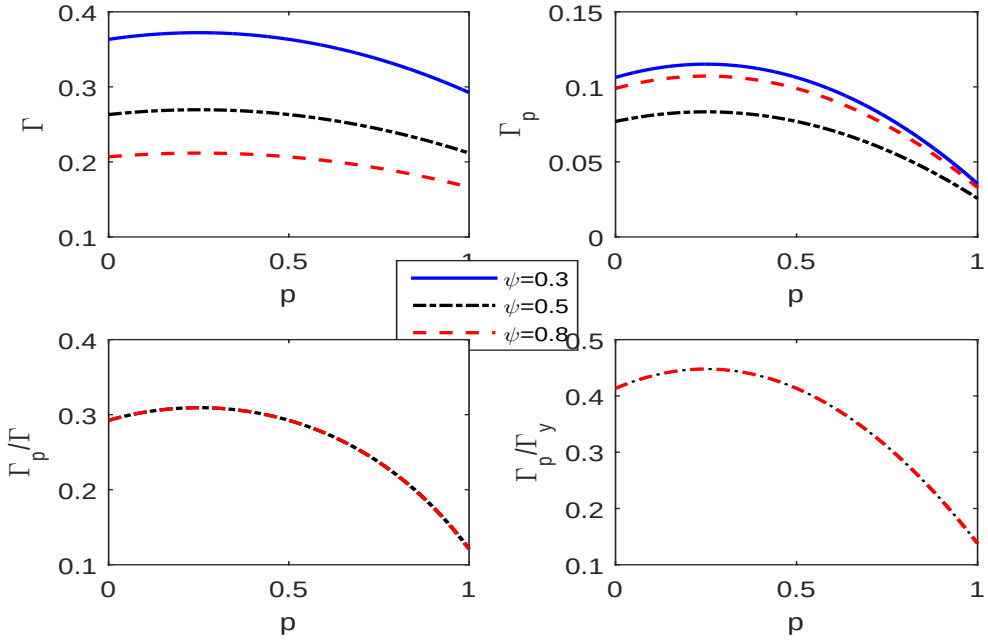

Figure 4: Effects of RB on Precautionary Saving

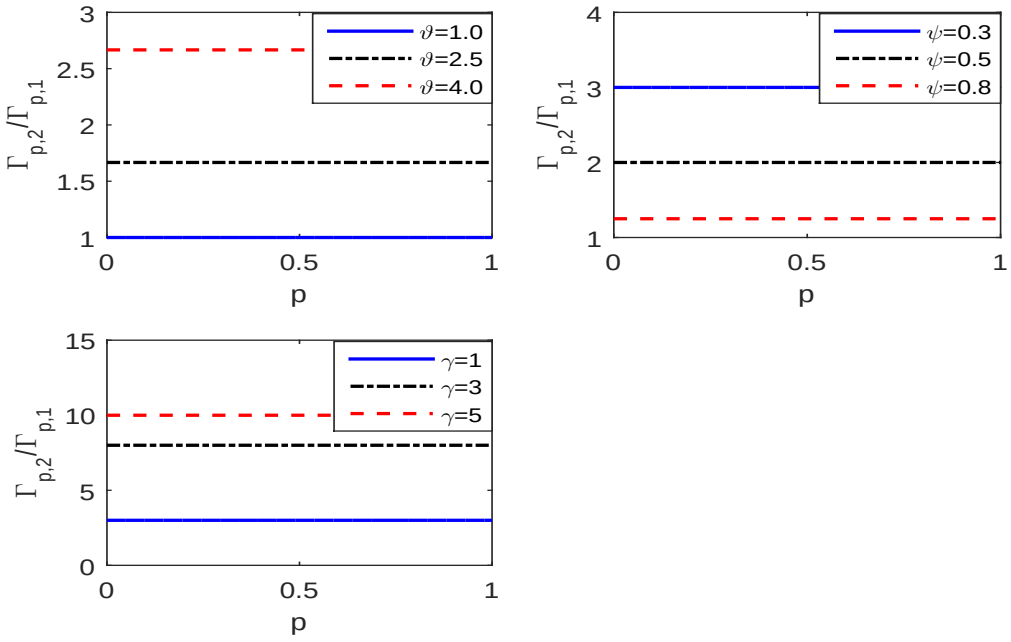

Figure 5: Relative Importance of RB in determining Learning-induced Precautionary Saving

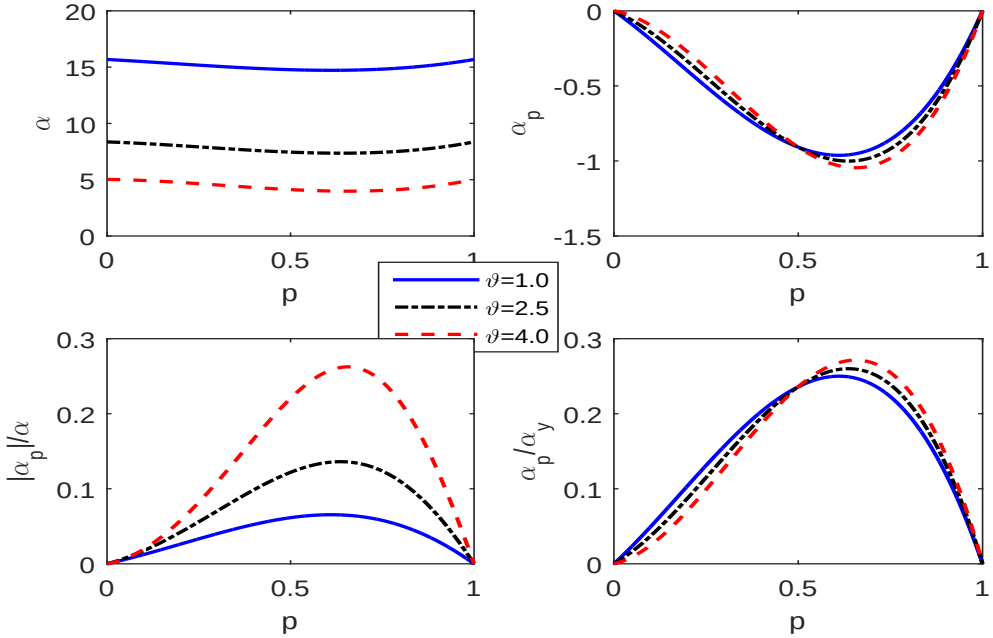

Figure 6: Effects of RB on Portfolio Choice ($\psi = 0.5$)

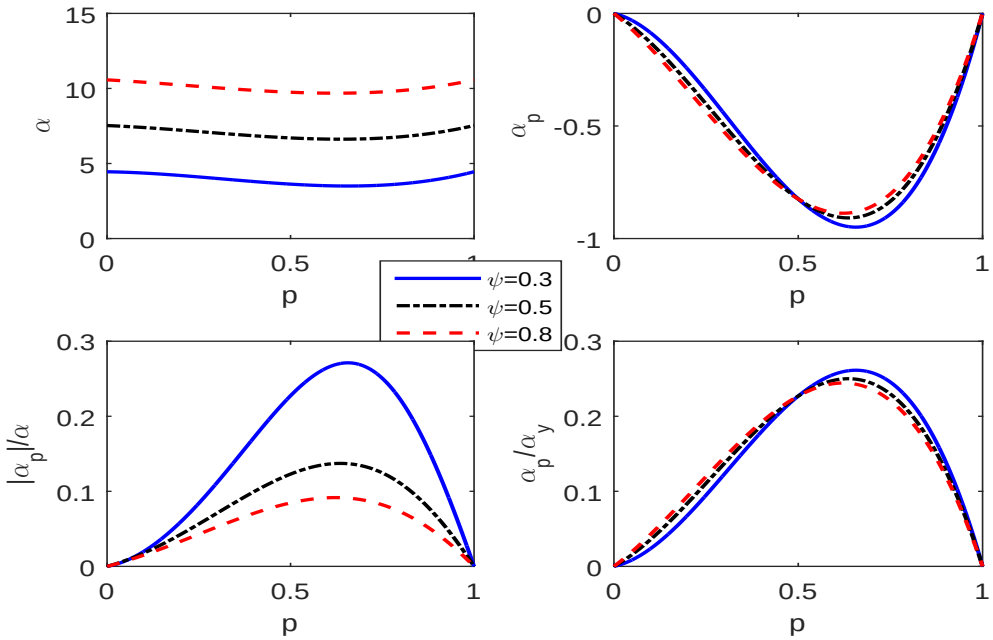

Figure 7: Effects of RB on Portfolio Choice ($\vartheta = 2$)

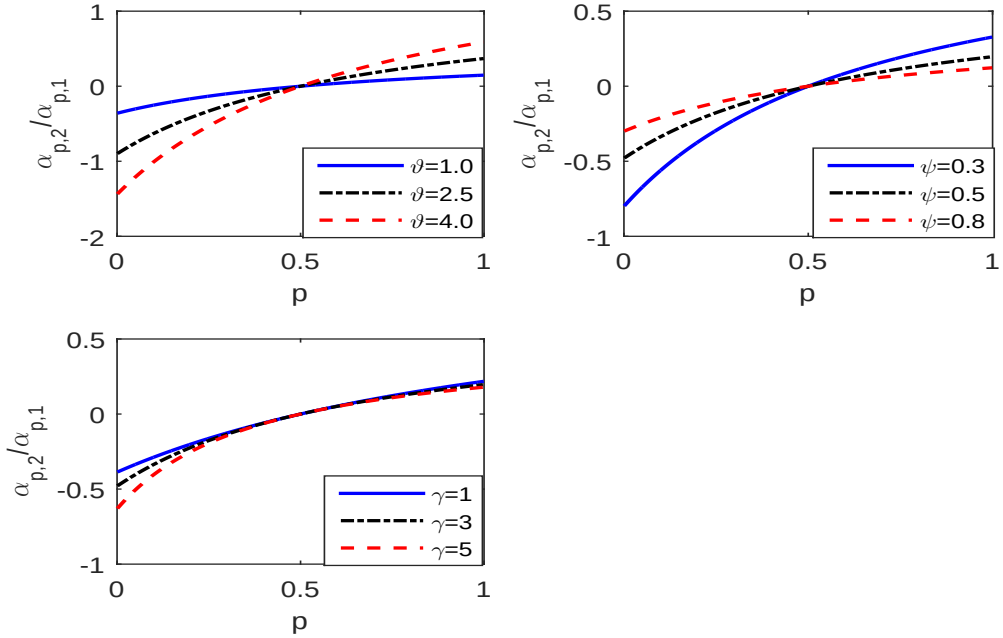

Figure 8: Relative Importance of RB in determining Learning-induced Portfolio Choice

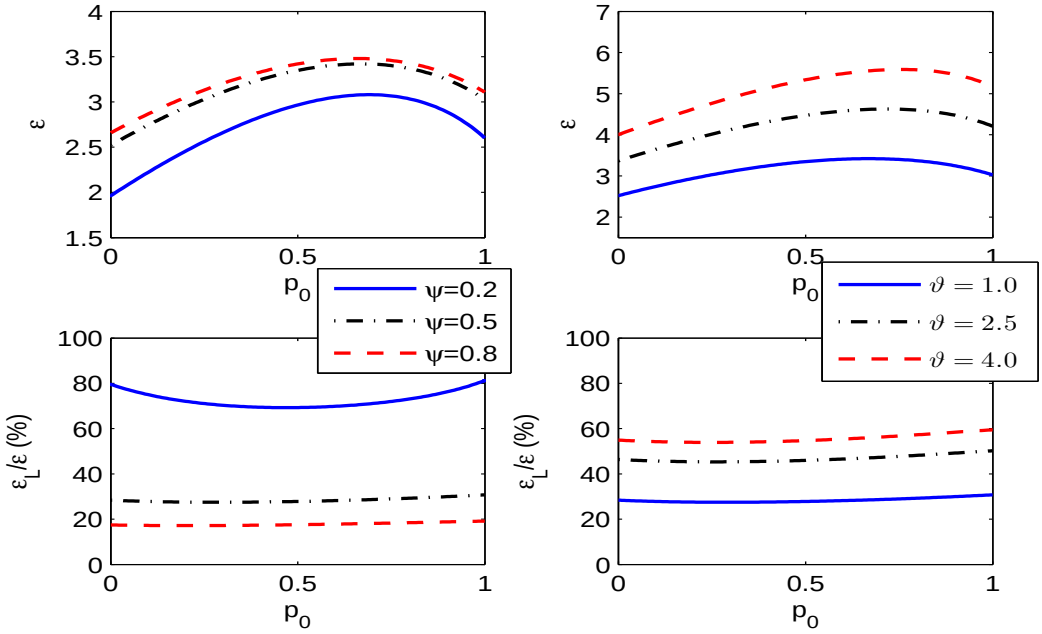

Figure 9: The Effects of Incomplete Information and Ambiguity on ε and $\varepsilon_L/\varepsilon$.

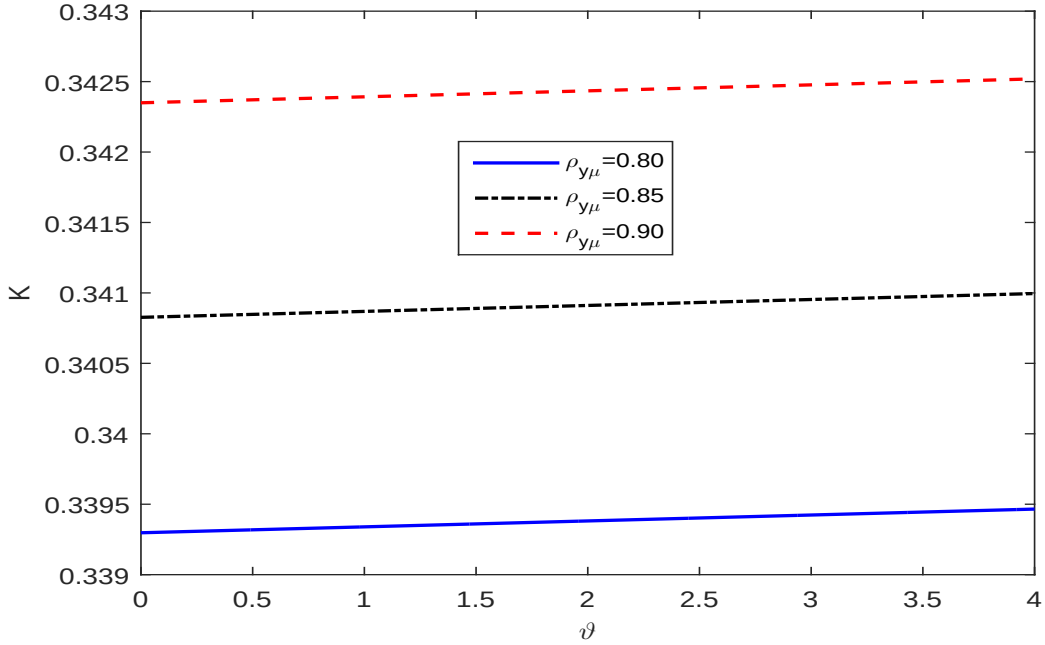

Figure 10: Effects of RB on the Robust Kalman Gain

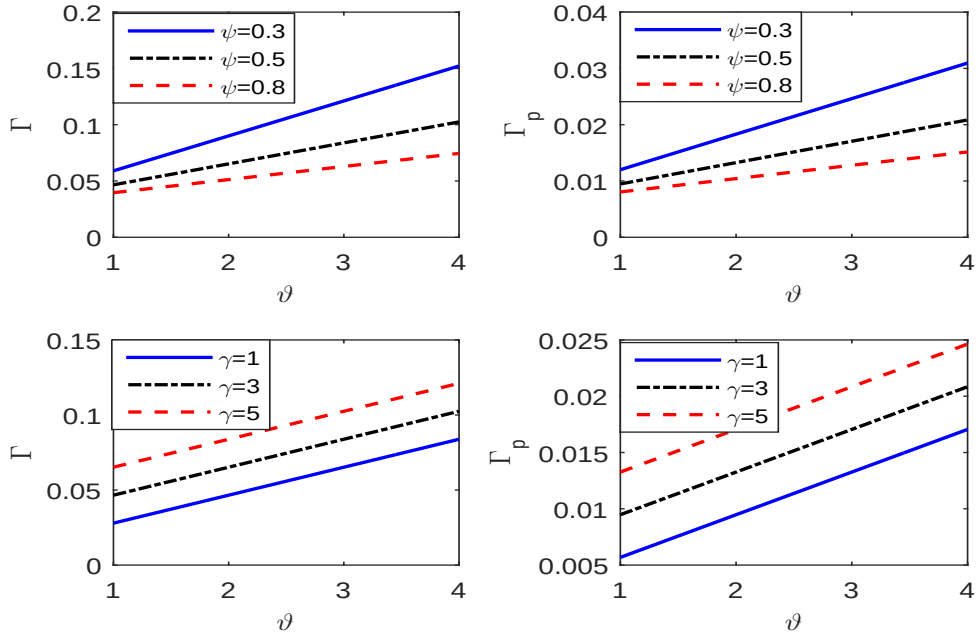

Figure 11: Effects of RB on Precautionary Saving for Different values of ρ_{ye}

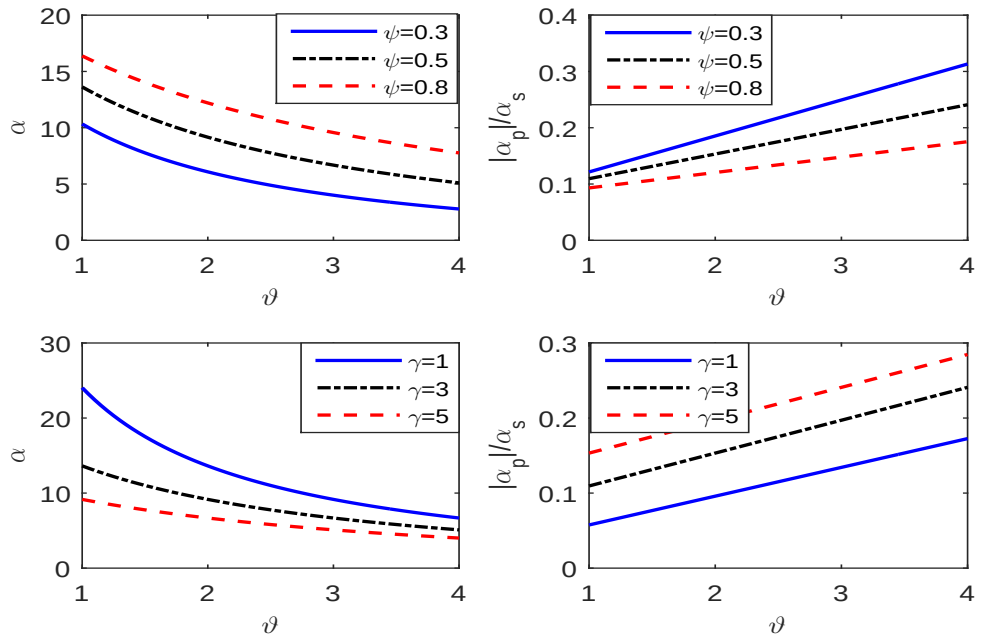

Figure 12: Effects of RB on Portfolio Choice for Different values of ρ_{ye}

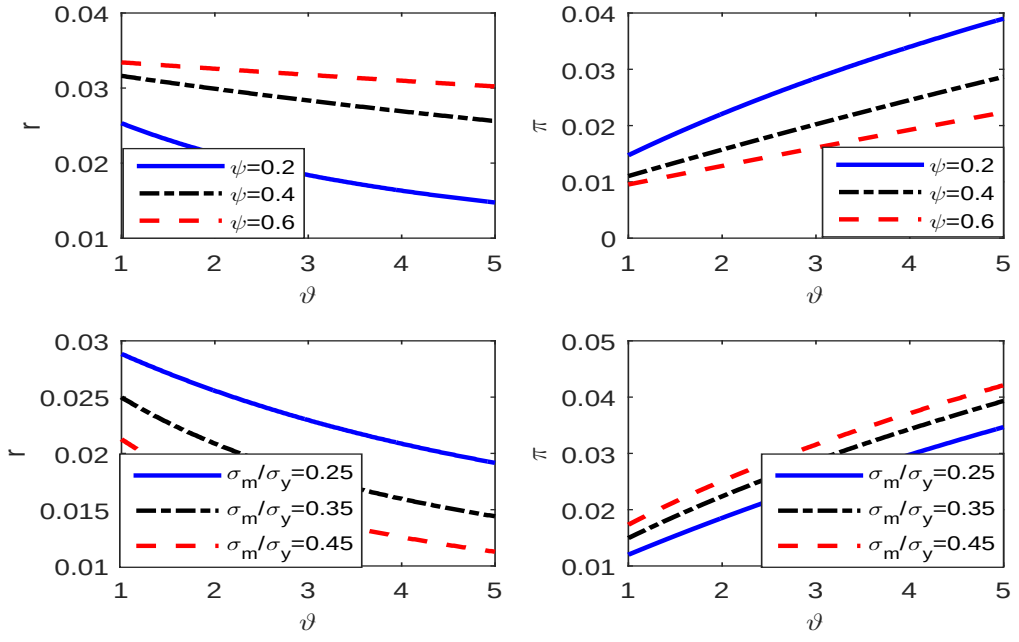

Figure 13: Effects of Ignorance on Equilibrium Interest Rate and Risk Premium

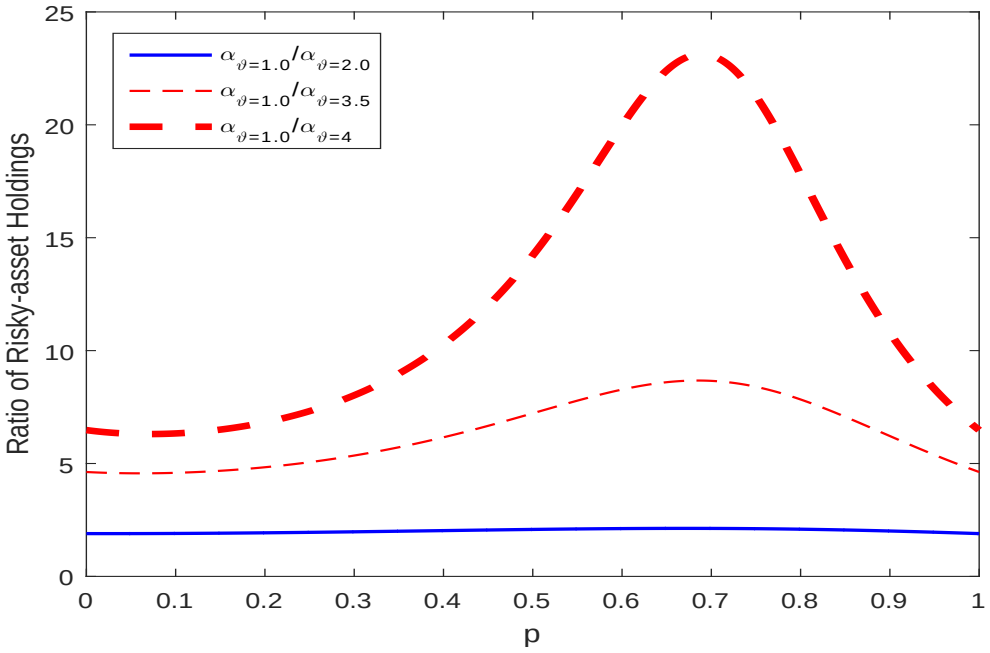

Figure 14: Interaction between MU and PU on Risky-asst Holding

Table 1: Number of households by education and income percentiles

Income Percentile	Less Than High School	High School and Some College	College and Above
$0-25$	$1,616$	$2,525$	479
$25-50$	728	$2,999$	893
$50-75$	366	$2,809$	$1,445$
$75-100$	163	$1,940$	$2,518$
Full Sample	$2,873$	$10,273$	$5,335$

Table 2: Mean stock values (in dollars) by education and income percentiles

Income Percentile	Less Than High School	High School and Some College	College and Above
$0-25$	$2,601.05$	$4,923.80$	$30,479.56$
$25-50$	$6,108.24$	$12,235.73$	$27,215.71$
$50-75$	$7,268.03$	$10,428.84$	$42,343.12$
$75-100$	$16,550.95$	$31,222.16$	$183,824.90$
Full Sample	$4,875.31$	$13,529.19$	$105,522.10$

Table 3: Mean stock values as a proportion of wealth (with and without home equity) by education and income percentiles

Income Percentile	Less Than High School		High School and Some College		College and Above	
	with h.e.	without h.e	with h.e.	without h.e	with h.e.	without h.e
$0-25$	0.89%	1.63%	2.07%	5.87%	6.29%	11.21%
$25-50$	2.03%	3.35%	4.06%	7.12%	8.60%	14.36%
$50-75$	2.03%	6.40%	4.06%	6.77%	9.41%	14.74%
$75-100$	1.59%	5.75%	5.43%	8.57%	12.06%	18.36%
Full Sample	1.42%	2.97%	3.69%	7.00%	10.25%	16.07%

Table 4: Risky-asset Holding (α): The Model' Predictions and Data

	Less Than High School	High School and Some College	College and Above
MU Parameter (ϑ)	3.92	3.30	1.00
PU Parameter (p)	0.50	0.50	1.00
Model's Prediction on α	1 (normalized)	2.8	21.4
Data	1 (normalized)	2.8	21.4