A Model of Financial Fragility

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Abstract

This paper presents a dynamic, stochastic game-theoretic model of financial fragility. The model has two essential features. First, interrelated portfolios and payment commitments forge financial linkages among agents. Second, iid shocks to investment projects' operations at a single date cause some projects to fail. Investors who experience losses from project failures reallocate their portfolios, thereby breaking some linkages. In the Pareto-efficient symmetric equilibrium studied, two related types of financial crisis can occur in response. One occurs gradually as defaults spread, causing even more links to break. An economy is more fragile ex post the more severe this financial crisis. The other type of crisis occurs instantaneously when forward-looking investors preemptively shift their wealth into a safe asset in anticipation of the contagion affecting them in the future. An economy is more fragile ex ante the earlier all of its linkages break from such a crisis. The paper also considers whether fragility is worse for larger economies.

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> Financial fragility is, to a large extent, an unavoidable consequence of a dynamic capitalistic economy. Its fundamental sources … cannot be eliminated by government intervention, and attempts to do so may create more instability than they prevent. [Calomiris, 1995, p. 254]

> [T]he Great Depression, like most other periods of severe unemployment, was produced by government mismanagement rather than by any inherent instability of the private economy. [Friedman, 1962, p. 38]

> If [someone] would only fully specify any one financial-fragility model …, perhaps we could think more clearly about the potential scope of the argument. [Melitz, 1982, p. 47]

I. Introduction

The concept of financial fragility dates back to Fisher (1933) and Keynes (1936), who theorized that the debt financing of investment can have destabilizing effects. Both economists' writings were motivated by their personal observations of the Great Depression and of numerous banking panics. In recent decades, Minsky (e.g., 1977) has advanced a slightly stronger version of the same idea—namely that modern capitalist economies are *inherently* fragile because of their heavy reliance on debt to finance investment. Although his efforts stimulated a large literature on financial fragility, that literature has lacked a fully coherent model that could provide some insight about the nature of fragility and its implications.[1] The goal of this paper is to fill that gap by presenting a model in which the concept of fragility is well defined and using it to explore the factors that determine an economy's fragility and whether fragility worsens as an economy increases in size.

Two features seem essential to any model of financial fragility. First, the economic environment must drive agents to take actions that forge links between their financial positions and the positions of others. Second, the environment must drive agents to take actions that break those links, in some cases completely, and in others only to a limited extent. The model presented in Section II minimally embodies these features.

To capture the first feature, the model is designed to give rise under certain conditions to interrelated portfolios and payment commitments that create financial linkages among agents.

[1] There is a considerable literature on financial crises, which will be reviewed at the end of this section. Discussions of fragility are either absent from it or purely informal.

There are two types of agents: investors and entrepreneurs, possibly many of each. Investors choose investment strategies—rules determining whether they hold their wealth in a safe, noninterest-earning asset or make risky loans to entrepreneurs at the market interest rate. Preferences and parameter values can be such that there exist equilibria in which investors choose portfolios that are linked in the sense that the return on an investor's portfolio depends on the portfolio allocations of other investors. These linkages collectively constitute the economy's financial structure. Because investors only know their own portfolios and not other investors', they do not know to whom they are linked.

Randomness enters the model through independent and identically distributed shocks to projects' operations. These shocks are allowed to occur only at a single date, and the economy's response to them is traced over time. The shocks cause some projects to fail, generating the second essential feature of the model, namely that some linkages come undone. When projects fail, the projects' entrepreneurs default on their loans, which causes some investors to suffer losses on their portfolios. In response, the affected investors reallocate their portfolios. They choose not to renew their loans, thereby breaking their financial links to other investors. Because of these portfolio reallocations, some entrepreneurs cannot fund the continued operation of their projects and thus default themselves. The process continues until a new steady state is reached, with no further defaults or portfolio reallocations.

Sections III and IV consider financial crises and fragility. A financial crisis is a breakdown of the economy's financial linkages, a collapse of all or part of the financial structure. The scenario described above is of one type of financial crisis that can occur in the model: actual defaults and losses spread as investors reallocate their portfolios in response to past losses incurred. These crises, studied in Section III, can arise if investors do not foresee the possibility of contagious defaults affecting them. Without foresight, fragility must be evaluated ex post, from the perspective of after the shocks have been realized, in terms of the severity of the crisis experienced. This approach to fragility seems natural and in fact is the one that has typically been taken. But the model teaches that the severity of the resulting crisis is randomly determined, so using an ex post approach will never yield a definitive conclusion about an economy's fragility. The next crisis realized can always be dramatically different from past crises.

In Section IV, the case in which investors foresee contagious defaults is discussed. When there is foresight, a second type of financial crisis is possible, one that occurs when all investors simultaneously shift to safer portfolios in anticipation of defaults possibly affecting them. Like the other type of crisis, this one is caused at root by fundamentals, namely the distribution of shocks, portfolio linkages, and the returns from various portfolios. This crisis also suggests evaluating fragility from an ex ante rather than an ex post perspective by asking how soon an economy's financial structure would collapse completely from such a crisis if shocks were to hit at some date.

Because one goal of this paper is to determine whether the economy is in some sense inherently fragile, attention is restricted to equilibria with foresight in which the economy has the best chance of surviving financial crises with its financial structure intact. In such equilibria, investors maintain diversified but risky portfolios as long as possible, given that they expect other diversified investors to remain so. Under the equilibrium strategies, investors remain diversified and linked if the amount of time that has passed does not exceed a state-dependent threshold. That threshold date is the first date at which the economy experiences a crisis caused by individuals simultaneously shifting to the safe asset. The earlier that date is, the more fragile is the economy. That date depends on the features of the environment that interact to determine which portfolios investors prefer: the utility function and degree of risk aversion, the discount factor, the rates of return on the various assets, the riskiness of the assets, the degree of diversification possible, and expectations about other investors' strategies. The use of an ex ante notion of fragility thus yields an unambiguous measure of fragility.

A related issue is how ex ante fragility depends on the size of the economy. A larger economy in the context of the model is one with more investment projects, entrepreneurs, and investors. As an economy increases in size, opportunities for portfolio diversification typically increase, which contributes to reduced fragility. But as portfolios become more diversified, they also become more interconnected, so the failure of a project somewhere in the economy can spread to affect a greater number of projects and investors. It is shown in Section IV that fragility can worsen as the economy increases in size beyond some point, holding fixed the degree of diversification. An implication of this finding is that studying the fragility of only small economies can be misleading. Section V modifies the economy of Section II to allow for greater diversification, holding fixed the degree of interconnectedness, and shows by way of an example that greater diversification reduces financial fragility. Section VI considers institutional responses to fragility, and Section VII presents some concluding remarks.

Since there is a large literature on financial crises, it is appropriate to discuss this paper's connection to that literature before moving on to describe the model more fully. Most of that literature does not allow for contagion. Papers within this group are primarily of one of three types. One type generates crises from herd behavior—agents copy other agents' actions because they think the others have better information (e.g., Banerjee 1992, Bikhchandani et al. 1992, and Chari and Kehoe 1997). While limited information also is critical to the crises in this paper, here agents do what maximizes their expected utility given that other agents behave in ways that make crises least likely. A second type of paper generates crises from asymmetric information between borrowers and lenders (e.g., Mishkin 1991). In that literature, borrowers are assumed to have private information about the investment projects they wish to operate, resulting in an adverse selection problem. In contrast, in this model expected and realized project returns are known to all agents. The only private information here concerns investors' portfolios: each investor knows his own portfolio but not those of other investors. With this limited knowledge, investors cannot determine to whom they are linked or their true exposure to contagion risk. A third type of paper generates financial crises from extraneous randomness, or sunspots. This literature has had considerable success in explaining the Great Depression and the more recent Mexican debt crisis (see Cooper and Corbae 1997 and Cole and Kehoe 1996, respectively). Fragility certainly could be studied in a model with sunspot-driven crises, but here the focus is on crises driven by fundamentals, as in Allen and Gale (1996), Atkeson and Ríos-Rull (1996), and Kiyotaki and Moore (1997a,b).[2]

There is also a growing literature that models contagion generally or in environments lacking financial factors. These models, known as local-interaction models, typically posit an exogenous graph in which each agent interacts with other agents positioned sufficiently close to him. Some types of behavior are shown to spread rapidly under certain behavioral rules (see, for example, Morris 1997 and the references contained therein). A few papers have used local-

[2] More specifically, the model in this paper assumes that shocks—presumably real shocks—to projects' operations cause projects to fail at the initial date, and that insufficient funding causes projects to cease operation at later dates. The financial crises associated with these events do not require that projects fail for these reasons. Rather, financial crises will arise in the model as long as project failures, no matter what their cause, induce portfolio reallocations that spread through portfolio linkages. Thus, the model could be modified so that sunspots, coordination failures, or what Federal Reserve Board Chairman Alan Greenspan has called "irrational exuberance" could generate the financial crises studied. See Lagunoff and Schreft (in progress) on the latter.

interaction models to study the role of production-based links in driving aggregate fluctuations. Bak et al. (1993) model a multi-sector economy with producers exogenously linked through supply relationships. Independent shocks to the demands for final goods affect the production of final goods and possibly of intermediate goods if intermediate-goods producers cannot fill orders from their inventories. Durlauf (1993) develops a dynamic, stochastic growth model consisting of many industries that can choose which of two production techniques to use. Each industry's current-period production function depends on the production techniques used in the previous period by an exogenously specified set of other industries, so industry-specific productivity shocks can spread over time across industries. In all the local-interaction models, however, agents are either myopic or boundedly rational; they only react once new behavior directly affects them. In contrast, this paper considers forward-looking and rational agents who anticipate the spread of defaults and so may take preemptive action to protect themselves.

Bridging these literatures are the few papers, such as the present one, that model contagious financial crises. Rochet and Tirole (1996), Fujiki, Green, and Yamazaki (1997), and Kiyotaki and Moore (1997a) are among the papers in this category. In all of these papers, financial linkages exist and can be broken by routine economic shocks that propagate through the linkages. Rochet and Tirole posit a set of unspecified exogenous linkages among banks engaged in interbank lending. The failure of a single bank due to a small liquidity shock can lead to the closure of all banks in the economy. Fujiki, Green, and Yamazaki focus on payment commitments that arise from transfers of goods among just three or four exogenously linked traders. Kiyotaki and Moore model an economy with many final- and intermediate-goods producers linked through their supply relationships and reliant on trade credit. Shocks to some firms' revenues can result in defaults that can propagate and cause a generalized recession.

This paper differs from the other papers modeling contagious financial crises in three key respects. First, in those papers, the agents are not anonymous: they know with whom they are dealing and thus to whom they are linked, at least directly. This paper, in contrast, models well-known entrepreneurs that issue debt in a perfectly competitive credit market and investors who are unknown to one another. The focus is on the financial linkages that arise endogenously despite anonymity. Second, in the alternative models of contagious financial crises, agents are either myopic or their decision problems are static. This paper, however, models investors who face genuinely dynamic decision problems and foresee the future implications of their current decisions. That is, the investors recognize the future consequences of current linkages. Finally, the other papers are silent on the subject of fragility, whereas this paper's goal is to define and characterize the fragility of an economy.

II. The Economy

The economy described here is very primitive, having only the features necessary to forge financial linkages among agents. Specifically, the economy drives agents to choose minimally diversified portfolios whose returns depend on the portfolios of other agents. Payment commitments arise out of the asset purchases associated with the portfolio choices. Idiosyncratic shocks to some assets result in some payment commitments going unfulfilled, which causes losses on portfolios consisting of those assets. Holders of those portfolios reallocate their wealth to regain an optimal asset mix, and in the process undo some financial linkages.

Despite the simplicity of the portfolio-choice problem, the model permits numerous financial structures—patterns of linkages—to form. The return on an agent's portfolio depends on the portfolio's contents and the agent's position in the financial structure. Conveniently, there are only a few types of positions that the agent can be in, so solving for his investment strategy is tractable. The remainder of this section describes the physical environment, characterizes the financial structures that can arise, and defines financial crises and financial fragility.

A. Agents, Preferences, Endowments, and Technologies

Time is discrete and continues forever, and at each date the economy is populated by k infinitely lived agents of each of two types: entrepreneurs and investors. At the initial date, $t = -1$, each entrepreneur is endowed with a risky investment project but no funds with which to operate it. Funds are indivisible objects called "dollars," which may be a form of money but need not be. Each investor has $2 that he wishes to invest to support consumption in future periods, and has access to a safe, noninterest-earning asset in which he can invest directly. An investor also has the option of making risky operating loans to entrepreneurs that offer the chance of a higher return.

More specifically, at each date an entrepreneur's project yields a random return of $R(I)$

dollars per dollar invested, where I denotes the total number of dollars invested. Each project can be operated only if it has sufficient funding. For simplicity, the critical level of funding—the level at which projects pay the maximum return per dollar, $\overline{R}$—is taken to be \$2. Projects that have less than two dollars invested in them pay a gross return per dollar of zero.[3] Once a project has been insufficiently funded, it becomes inoperable at all future dates. When a project is overfunded, with more than two dollars invested in it, decreasing returns are realized and the project yields a return per dollar of $2\overline{R}/I$.

At date 0 only, there is a second way by which a project can become inoperable. Independently and identically distributed shocks can hit projects, causing them to fail and pay a zero return. A shock hits a project with probability p, $p \in (0,1)$. Once a project fails, it is forever inoperable. If a project succeeds at date 0, it pays a return that depends on the amount invested, as described above.

In summary, then, a project's return per dollar, assuming the project has not previously ceased operation, satisfies

$$R(I) = \begin{cases} 0 \text{ with probability } p \text{ and } \dfrac{2\overline{R}}{I} \text{ with probability } 1-p & \text{at } t = 0 \text{ if } I \geq 2, \\ \dfrac{2\overline{R}}{I} & \text{at } t > 0 \text{ if } I \geq 2, \\ 0 & \text{at } t \geq 0 \text{ if } I < 2, \end{cases}$$

where $\overline{R} > 1$ and is bounded above.[4] The assumption of an upper bound is made for simplicity, not out of necessity. The upper bound is set to ensure that (1) no portfolio can ever yield a full dollar in interest and thus increase the wealth available for investment, and (2) an investor who sustains a loss of any magnitude can reinvest in at most one project. Given that investors are initially endowed with \$2 and that dollars are indivisible, the first condition implies that $\overline{R} < 1.5$ so that no one ever invests in more than two projects. The second condition, which requires a tighter upper

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[3] Technologies with a minimum scale of operation have long been used in the literature on financial intermediation. See Diamond (1984).

[4] The assumption that the shocks are iid differs from that often made in models of financial crises; typically shocks are perfectly correlated (e.g., Allen and Gale 1996). Since the objective here is to assess whether economies are inherently fragile, the assumption that shocks are iid is preferable because it gives the economy its best chance of avoiding being labeled "fragile."

bound, rules out an unusual situation, discussed in Appendix A, for notational simplicity.[5]

Each investor has preferences defined over wealth consumed, c_t, at dates $t \geq 0$ but not date -1. He chooses at each date how much of his wealth to consume immediately and how much to carry into the next period to fund investment in a portfolio of safe and risky assets. The return on this portfolio is used to fund future consumption. The economy's single safe asset always yields a return of one dollar per dollar invested. The risky assets are loans to entrepreneurs, who are assumed to be price takers in the loan market. Without loss of generality, all loans are assumed to be for one period.[6]

The timing of economic activity, illustrated in Figure 1 below, is as follows. At date -1, investors consume nothing and simply choose a portfolio of assets to purchase at the start of date 0.

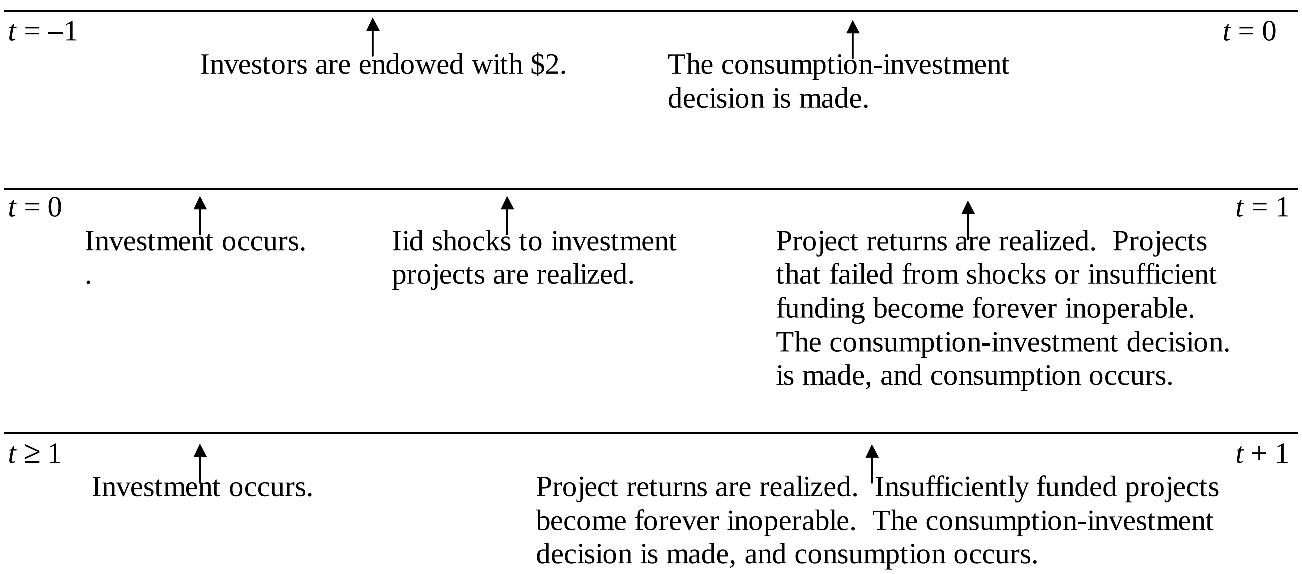

Figure 1—The Timing of Economic Activity

[5] The entrepreneurs are uninteresting in this model. They simply take funds lent to them and operate projects. The model works identically if the entrepreneurs are omitted and investors fund and operate projects directly. The reason for having entrepreneurs in the model is to introduce indirect finance, debt, and default, which better matches the stories underlying the financial-fragility literature. However, the nonessential nature of the entrepreneurs indicates that debt and default are not essential to fragility.

[6] Nothing in the physical environment requires that loans be for only one period. Allowing long-term loans would not affect the model's results because all entrepreneurs' projects are identical ex ante, so investors who choose to lend for multiple periods have no incentive to change the set of entrepreneurs to whom they lend.

A real-world analog to the financial market modeled here is the U.S. commercial paper market. In that market, borrowers are well known and issue low-risk, large denomination (typical face values are in multiples of $1 million), and short-term debt (the average maturity is 30 to 35 days) that is often continuously rolled over to finance current transactions such as operating expenses.

After the purchases have been completed, the iid shocks to investment projects are realized, causing some projects to fail. If an entrepreneur's project fails because a shock hit it or from insufficient funding, the entrepreneur defaults on his loan payments and the investors who lent to him are repaid nothing. After returns are realized, both at date 0 and at all later dates, investors choose how to divide their remaining wealth between consumption in the current period and an investment portfolio that they purchase at the beginning of the next date.

Based on this timing of economic activity, an investor's decision problem can be written formally as follows. A portfolio at date $t \geq 0$ is a triple $\left(a_{ht}, a_{jt}, a_{st}\right)$, where a_{ht} denotes dollars invested in risky project h at t, a_{jt} denotes dollars invested in risky project j, $j \neq h$, and a_{st} denotes dollars invested in the safe asset. An investor who chooses to invest \$2 can hold any of the following portfolios: a diversified loan portfolio ((1,1,0)), an undiversified loan portfolio ((2,0,0) or (0,2,0)), a part-safe-part-risky portfolio ((1,0,1), (0,1,1), (1,0,0), or (0,1,0) since only part of the investor's total wealth is at risk with these portfolios), or a safe portfolio ((0,0,2), (0,0,1), or (0,0,0) since none of the investor's wealth is at risk with these portfolios).[7] An investor who chooses to invest \$1 can hold an undiversified loan portfolio ((1,0,0) or (0,1,0)) or a safe portfolio ((0,0,1) or (0,0,0)). The gross return on portfolio $\left(a_{ht}, a_{jt}, a_{st}\right)$, which is realized at t, is $r\left(a_{ht}, a_{jt}, a_{st}\right)$. At each date $t \geq -1$ then, an investor with post-return wealth y_t, either from an endowment (at date -1 only) or from the realized return on his portfolio, chooses a sequence $\left\{\left(a_{h\,t+1}, a_{j\,t+1}, a_{s\,t+1}\right), c_t\right\}$ to solve

$$\max E_t \sum_{h=t}^{\infty} b^{h-t} u(c_h), \quad 0 < b < 1 \text{ and } h \geq 0,$$

subject to

$$c_h + a_{h\,h+1} + a_{j\,h+1} + a_{s\,h+1} \leq y_h \quad \text{for } h \geq -1,$$
$$y_{h+1} = y_h - c_h - \left(a_{h,h+1} + a_{j,h+1} + a_{s,h+1}\right) + r\left(a_{h,h+1}, a_{j,h+1}, a_{s,h+1}\right) \quad \text{for } h \geq -1,$$
$$a_{h\,h+1}, a_{j\,h+1}, a_{s\,h+1} \in \{0, \$1, \$2\}, \;\; c_h \geq 0, \;\; y_{-1} = \$2,$$

where $u(\cdot)$ is an increasing, strictly concave, and time-separable von Neumann-Morgenstern utility function with $u(0) = 0$. The first constraint simply states that total expenditures on consumption and investment cannot exceed available wealth. The second condition describes the evolution of post-

[7] The assumption that dollars are indivisible is equivalent to an assumption that investments must be made in \$1 increments.

return wealth and reflects the fact that investors can either consume or reinvest their portfolio returns. Wealth consumed is no longer available for investment. The final set of conditions states that investments must be made in increments of \$1, that consumption must be nonnegative, and that an investor's initial wealth is \$2.

A comment is in order about the assumption that dollars are indivisible since, given the fixed wealth endowment, it limits loan size and thus the degree of diversification. It is well known that investors are better off when more fully diversified, but that, in practice, investors' portfolios exhibit only limited diversification. Less is known about the costs of diversification.[8] In the model, fixing the extent to which investors can diversify also fixes the benefits investors can achieve through diversification, which allows attention to be focused on diversification's costs. Those costs are associated with the linkages among portfolios and the economy's fragility. One can ask, then, how the costs of holding diversified portfolios change as an economy increases in size. Sections III and IV take this approach. Section V considers the effect of greater diversification by increasing the wealth endowment while maintaining the indivisibility of dollars and holding fixed the degree of interconnectedness.

B. Portfolio Choices, Chains, and Chain Structures

The solution to the investor's portfolio-choice problem depends on the functional form of utility and the values of the model's parameters, and on certain features of an equilibrium, which has yet to be defined. Only certain portfolio choices are of interest because, as discussed in the introduction, a model of financial fragility requires first that investors' portfolios be linked at some point in time, and second that investors break those links under some conditions at a later date. To focus on the concept of fragility in what follows, it is conjectured that an equilibrium exists with these two properties. This is equivalent to conjecturing that there are ranges of the parameter space for which, given preferences and the equilibrium concept, an equilibrium exists in which investors rank certain portfolios in certain ways. The remainder of this subsection formally states each property and describes the role the property plays in generating or breaking linkages. In Sections III, IV, and Appendix A, the investor's equilibrium portfolio choices are determined in all other

[8] Krasa and Villamil (1992) show that limited diversification is optimal when there are monitoring or similar costs associated with holding diversified portfolios.

circumstances. Fragility is defined and characterized when investors act with and without foresight about the implications of the linkages. Finally, Appendix B shows that there exist preferences and an open set of the parameter space for which the conjectured properties of an equilibrium are indeed consistent with the equilibrium considered.

The first conjectured property is

PROPERTY 1. At $t = -1$, an investor with \$2 prefers a *maximum-return diversified loan portfolio*, a portfolio (1,1,0) with both loans made to entrepreneurs who end up receiving full funding, to any other feasible portfolio.

This property accomplishes two objectives: It defines a maximum-return diversified loan portfolio, and it implies that investors choose such portfolios at $t = -1$.[9] It follows that the portfolio linkages at the beginning of date 0 can be represented with closed chains. A *closed chain* is a set of entrepreneurs (and thus investment projects) and investors such that each project is fully funded, each investor is fully invested and diversified, and investor portfolios are all linked directly or indirectly. Figure 2 below illustrates a closed chain.

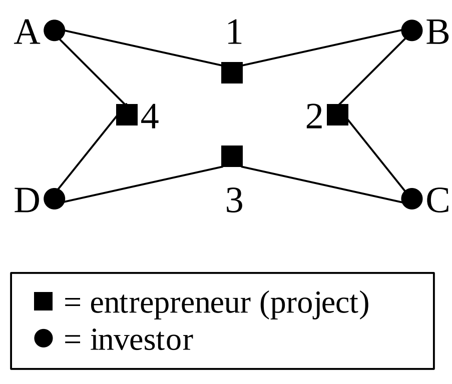

Figure 2—A Closed Chain

More precisely, the figure shows a closed *four-link* chain because the chain links each investor either directly or indirectly to four projects. Investor A, for example, is directly linked to

[9] It is not tractable to consider economies in which entrepreneurs differ at date −1 in the amount lent to them. Economies in which all entrepreneurs borrow the same amount in excess of \$2 differ from the economies studied here only in that the return on projects is lower and thus that investors' returns and consumption are lower.

the two investors nearest to him, investors B and D, because his portfolio contains loans to finance projects 1 and 4, projects to which B and D, respectively, also lent funds. He also is indirectly linked to the remaining investor, investor C, because although his portfolio has no assets in common with C's, C's portfolio has assets in common with B's and D's.

The iid shocks that occur at date 0 turn closed chains into open chains. An *open chain* is a set of entrepreneurs (projects) and investors such that at least one project is fully funded, each

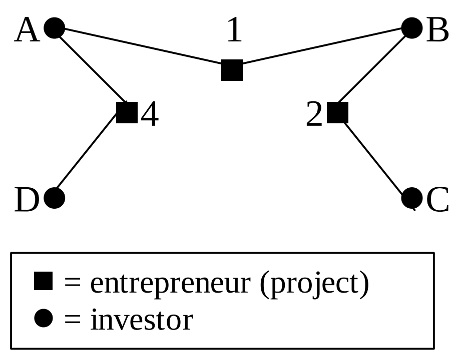

Figure 3—An Open Chain

investor is fully invested but not necessarily diversified, and the investors' portfolios are all linked directly or indirectly. Figure 3 above depicts the open three-link chain that results when project 3 in Figure 2 fails.

The second conjectured property of an equilibrium addresses the implications of a closed chain becoming open. That property is

PROPERTY 2. At each $t \geq 0$, an investor who has lost \$1 prefers not to reinvest in a risky asset.

Property 2 states that after returns are realized at any date $t \geq 0$, investors who find themselves with at most \$1 to reinvest will, at the next opportunity, either immediately consume their entire wealth or consume their interest and reinvest \$1 in the safe asset for one period. Consequently, if, say, project 3 in Figure 2 fails from a shock at the initial date, then its entrepreneur defaults on his loans from D and C, failing to repay even the principal of the loan. The losses to D and C lead them, given Property 2, to shift out of risky assets and consume their remaining wealth.[10] The remaining

[10] This demand for a safer portfolio is reminiscent of Keynes' (1937) liquidity preference.

entrepreneurs to whom D and C had made loans, those operating projects 4 and 2, respectively, then find their projects insufficiently funded. Since insufficiently funded projects (those run with less than $2) yield a return of zero, the entrepreneurs operating projects 4 and 2 default on the loans they received. This means that investors A and B incur losses solely as a result of the portfolio reallocations of investors D and C. Their losses drive them to reallocate their own portfolios, leaving project 1, to which they had both lent funds, with insufficient funding.

This example illustrates how the returns to each investor in a chain are affected by the portfolio allocations of all other investors in the chain. It also shows that the linkages in an open chain break over time as investors rationally adjust their portfolios in response to actual defaults experienced. This process results in open chains disintegrating completely over time.

The concept of chains leads naturally to the concept of a chain structure for the economy. The economy's *chain structure* at any point in time is the combination of open and closed chains that reflects the prevailing portfolio linkages. That is, it is a partitioning of investors and entrepreneurs (with their projects) into chains. By Property 1, the *initial chain structure*, that which exists once investment occurs at the start of date 0, consists *solely* of closed chains. In an economy with four entrepreneurs and four investors, for example, two initial chain structures are possible. One consists of a single closed chain in which all investors are linked to all projects, as shown in Figure 2. The other consists of two closed chains, each with two investors linked to two projects, as shown in Figure 4 below. Any other structures must have at least one degenerate chain with one investor linked to a single project. Such structures are inconsistent with optimizing initial portfolios because the utility function is strictly concave.

The process by which the initial chain structure arises is not modeled here. To model such a process requires a theory of why some investors might choose to invest in one project, while others might choose to invest in another. Such a theory is not now available. The theory one ultimately uses, however, will give rise to a prior that investors use in calculating their expected utilities from various portfolios. Only one of this paper's results—namely Proposition 2—depends on the properties of the prior.[11]

[11] Some earlier versions of this paper assumed a particular prior. That prior is used here only in Section IV as an illustrative example.

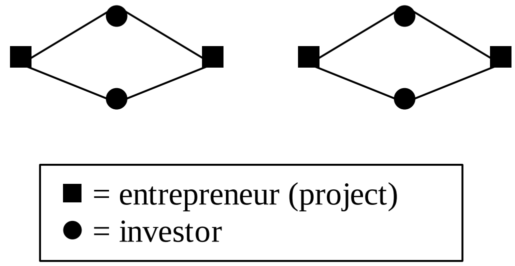

Figure 4—An Initial Chain Structure for the k = 4 Economy

At any date after the shocks are realized, the economy's chain structure is a mix of closed and open chains. The mix depends on the shocks realized at the initial date and the strategic portfolio choices of investors at subsequent dates.

C. Information and Communication

The information structure is a particularly important feature of the economy, as will become obvious later in the paper. All agents in the economy know k, which represents the number of projects that exists initially, and that the shocks at date 0 are iid. They also know the function determining project returns, $R(I)$. To capture the notion of large, anonymous economies, however, portfolio choices are assumed to be private information: investors know their own portfolio allocations, but not those of other agents. That is, they know to whom they lent funds but not to whom other investors lent funds. In particular, they do not know the identities of the others who made loans to the entrepreneurs to whom they lent funds. Finally, they know the realized returns on their own investment projects at all dates.

In addition to having limited information, agents also are assumed to have limited ability to communicate and thus to overcome the information restrictions. Limited communication ensures that agents remain anonymous and cannot join with other agents to share risk or prevent defaults.

An implication of the informational features of the economy is that investors do not know the chain structure of the economy, which type of chain—closed or open—they are in, or their location within their chain. In particular, if they are in an open chain, they do not know how close they are to the endpoints and thus to an impending default.[12]

[12] The information problem facing investors when allocating their portfolios is similar to that facing individuals when choosing sexual partners when sex involves the risk of catching a sexually transmitted disease. Individuals may or may

D. Financial Crises and Fragility: An Introduction

The model described above suggests a particular definition of financial fragility:

DEFINITION. An economy exhibits *financial fragility* if it possesses a propagation mechanism that allows small exogenous shocks at the initial date to generate financial crises that have large-scale effects on the financial structure and thus on real activity.

By this definition, fragility is a matter of degree. It remains to explore the financial crises that can occur and to characterize the nature and degree of fragility associated with them.

A financial crisis in the model is a breakdown of the economy's financial linkages, a collapse of all or part of the chain structure. To evaluate the fragility associated with the realization of such crises, this paper looks at how the shocks at date 0 work their way through the economy until a steady state is reached. In a steady state, all investors' portfolio allocations remain unchanged, the projects in operation and their returns remain unchanged, and thus investors' wealth and period consumption remain unchanged. The larger the share of the chain structure that collapses during the transition to the new steady state, the more severe the crisis initiated by the shocks. An economy that experiences a complete collapse of its chain structure has no portfolio links intact in the new steady state. In contrast, an economy that experiences only a partial collapse has some closed chains remaining in the new steady state, but fewer than at the initial date. The model generates two nonexclusive types of financial crises, and two related ways to characterize fragility, depending on whether investors can foresee the possibility of defaults spreading to them.

III. Fragility without Foresight about Contagion

The simplest approach to assessing fragility involves assuming that investors do not foresee contagious defaults. Without foresight, investors have no reason to reallocate their portfolios at any

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not know their current and past sexual partners, and they have at most limited accurate information about their partners' partners. Thus, they do not know the other people to whom they are connected, either directly or indirectly, through a string of past sexual encounters.

date after the shocks hit unless they personally experience a default. Specifically, there exists an equilibrium with subjectively rational investors in which investors have incorrect beliefs about their exposure to contagious defaults. These investors wait, like sitting ducks, until defaults spread to them. The outcome is one type of financial crisis: an *actual-default crisis*. As its name suggests, this crisis arises from investors rationally reallocating their portfolios in response to losses they have incurred from actual defaults.

The discussion in Section II of the unraveling of open chains was an analysis of an actual-default crisis. Any shocks that hit a chain cause investor losses, which lead to rational portfolio reallocations, which lead to some projects having insufficient funding, further defaults and investor losses, and further portfolio reallocations. Even if all investors continue to hold diversified portfolios until they actually experience a default, any chain hit by a shock at the initial date collapses eventually.

The actual-default crisis suggests one characterization of fragility. An economy is more fragile *ex post* the more severe the actual-default crisis that occurred in response to shocks that were realized. This characterization of fragility seems to be the one economists have used historically. In the financial-fragility literature, an economy's fragility is taken to depend on the number and severity of crises the economy experienced in the past. This literature has yielded no clear conclusions about fragility because there is no consensus about what constitutes a crisis or what constitutes a severe crisis (see, for example, Kindleberger 1989 and Schwartz 1986). The analysis above suggests that no clear conclusions will ever be reached by analyzing ex post fragility because the actual-default crises that generate that fragility are purely random events. Their likelihood depends on the economy's initial chain structure and the number and distribution of shocks realized, all of which are determined randomly.

If, for example, the economy initially consists of a single closed *k*-link chain, then any shocks that hit the economy must hit that chain, causing all its linkages to ultimately break. How fast the chain collapses depends on the number and distribution of shocks. If there are few shocks and they hit adjacent projects, the chain survives longer than if there are many shocks uniformly distributed around the chain.

Alternatively, a chain structure consisting of many chains, each with two people and two projects, as in Figure 4, is the least susceptible to actual-default crises. It allows maximum diversification with minimum risk of defaults spreading. With this structure, the investors in any chain hit by a shock immediately become endpoint investors, and their chain collapses. But investors in chains not hit by shocks are forever safe from actual-default collapse. As a result, the best-case scenario with this chain structure is when there are few shocks and they hit a few chains. The worst-case scenario is when there are many shocks distributed uniformly across chains. Thus, this chain structure gives the economy its best chance of surviving with some of its linkages intact, assuming of course that investors continue to hold diversified portfolios until they suffer losses from actual defaults.

IV. Fragility with Foresight about Contagion

The analysis of the no-foresight case naturally leads one to wonder what would happen if investors could foresee the threat of contagious defaults. With foresight, a second type of crisis—an *anticipated-default crisis*—becomes possible in addition to the actual-default crisis. In an anticipated-default crisis, investors simultaneously shift to safer portfolios (ones with fewer funds exposed to contagion risk) as protection against loss from expected defaults, even if they have not experienced a default themselves and do not know for sure that defaults are occurring. Since all diversified investors have the same information at t, if it is rational for one to shift to a safer portfolio, then it is rational for them all to do so. The implication is that an anticipated-default crisis involves an instantaneous collapse of all remaining chains, whether open or closed.[13]

Unlike actual-default crises, anticipated-default crises occur by choice, not chance. Nevertheless, actual- and anticipated-default crises are related and endogenous equilibrium outcomes. An actual-default crisis leads to the complete collapse of an economy's chain structure *only* if every chain is hit by at least one shock at the initial date *and* if investors strategically choose to hold diversified loan portfolios long enough for all chains to collapse.

Anticipated-default crises suggest a second characterization of fragility. An economy is

[13] The analogy to the U.S. commercial paper market discussed in footnote 6 extends to this discussion of anticipated-default crises. The default in 1970 of Penn Central, a major issuer of commercial paper, led to a period of severe illiquidity in the commercial paper market, during which time some issuers found themselves unable to roll over new paper and the volume of commercial paper outstanding decreased dramatically. This crisis can be viewed as a rational response by investors who were unable initially to determine whether Penn Central's default was due to firm-specific factors or to factors affecting issuers more generally, such as the ensuing recession, and thus symbolic of more defaults to come. See Calomiris (1993) for an extended discussion of the Penn Central episode.

more fragile *ex ante* the sooner its chain structure is expected to collapse if shocks hit at some date. Although ex ante fragility is not the notion of fragility that instinctively comes to mind or that has been used historically, it seems the more appropriate notion. When speculating about an economy's fragility, one is really asking from the perspective of the current date what would happen if shocks were to hit the economy at some later date. The model studied here suggests that the only clear-cut answer to that question comes from looking at when an anticipated-default crisis occurs, since such crises are not random events. The sooner such a crisis occurs, the more fragile is the economy.

A. The Problem with Modeling Foresight

The existing literature on contagion (e.g., Morris 1997) has not allowed agents to have foresight. It turns out that modeling foresight regarding contagion is very difficult. The reason is that if investors have too much information about the current and past state of the economy, they can figure out too much about the chain structure and distribution of shocks and will condition on this additional information in calculating the expected utilities from various portfolios. Such calculations appear to be intractable even for fairly small economies.

This particular problem arises in this model if the state of the economy is taken to be the initial number of projects (k) and the number of projects that remain at the beginning of each subsequent date up through the present. For example, if an investor knows that $k = 10$, that it is currently date 2, after returns have been realized, and that eight projects remained after returns were realized at date 0, six remained at date 1, and six currently remain, then he can figure out much about his exposure to contagious defaults. Specifically, he can determine that only two projects were hit by shocks at date 0, that two projects failed because of contagion at date 1, and that the contagion was complete by date 2. This is consistent with two possible scenarios. Either two adjacent projects were hit by shocks in a four-link closed chain at date 0, or one project in each of two two-link chains failed at date 0. Either way, investors who have not yet incurred losses know that they are safe forever from contagion.

Even if the state is taken to consist of the initial number of projects and the number of projects currently remaining, but not the entire historical path of projects remaining, an investor can rule out many possibilities. For example, if an investor knows that $k = 6$, that it is currently date 1, after returns have been realized, and that four projects remain, then he knows that at $t = 0$ either one or two projects were hit by shocks in a single closed two-link chain. The reason is that if any larger chain had been hit by even one shock at date 0, then two projects would have failed at date 1 from contagion, leaving three or fewer projects. Thus, the investor knows that any contagion that had started is over and that the initial chain structure had to have contained at least one closed two-link chain.

For simplicity, then, this paper takes the state of the economy to be simply the number of projects at the initial date, k. An investor knows the state and the current date. In the context of this model, this assumption is reasonable because an investor essentially chooses a portfolio of risky assets once, at date –1. After returns are realized at all subsequent dates, if he has not yet incurred a loss, he either maintains his existing diversified loan portfolio, which has paid him the maximum return thus far, or shifts his funds to a safer portfolio. Even with this simple treatment of the economy's state, investors can rule out certain outcomes (e.g., they know that they can never be in an open six-link chain if only four projects existed initially), but it is tractable to condition on such information.

B. An Equilibrium with Foresight

In addition to knowing the state of the economy, an individual investor knows s_t, the post-return state of his portfolio, that prevailing after returns are realized and when he has to make his consumption and portfolio-allocation decisions. That is, he knows at each date, after realizing his returns, whether he has incurred a loss of any or all of his initial \$2 endowment. His state s_t is from the set $\{W_t, H_t, Z_t\}$, where W_t is the state in which he still has his whole initial endowment, H_t is the state in which he has half of his initial endowment because he has incurred a \$1 loss, and Z_t is the state in which he has zero dollars left. Based on the state of the economy (k) and the state of the investor's portfolio, the investor chooses an action a regarding his portfolio, where a is from the set $\{D, U, P, S\}$. Action D is the choice to continue holding a diversified portfolio at the beginning of the next date. Action U is the choice to deviate to an undiversified portfolio; action P, a part-safe-part-risky portfolio; and action S, a safe portfolio (Section II-A describes each of these portfolios). With this specification of actions, a strategy for investor i in the recursive game that is played at dates $t \geq 0$ may be defined as a sequence $f^i \equiv \{f_t^i\}$, where if $f_t^i(s_t, k) = a$, the investor, given states s and k, chooses at t to take action a at $t + 1$. A strategy profile f is a list $(f^1, \ldots, f^k)$, and $\tilde{f}^{-i} \equiv (\tilde{f}^j)_{j \neq i}$ is investor i's forecast of other investors' strategies f^j for all $j \neq i$. Finally, the post-return expected lifetime utility at t from strategy f^i and forecast $\tilde{f}^i$, conditional on i's being in state s_t and the economy's being in state k, is denoted $V_t(f^i, \tilde{f}^{-i} | s_t, k)$. The following definition may now be stated:

DEFINITION. A *symmetric equilibrium* is a strategy profile f^* such that, for each pair of investors i and j, $f^{i*} = f^{j*}$, and f^{i*} maximizes $V_t(f^i, \tilde{f}^{-i} | s_t, k)$ for all s_t and for all $t \geq 0$, given the forecast $\tilde{f}^{-i}$.

A few comments about this definition are in order. First, there may exist asymmetric equilibria—equilibria in which some investors behave differently than others in some states. But because all investors have the same preferences and information in each state, any two investors in the same state s who differ in their strategies f_t^i at some date t must be indifferent between remaining diversified and shifting to safer portfolios. This indifference will not generally hold except for very special parameters. In this sense, such equilibria are knife-edged.

Second, there is a class of symmetric equilibria with the property that each investor shifts to holding a safer portfolio at some date t solely because he expects others to do the same. Such sunspot equilibria represent coordination failures driven by strategic behavior analogous to that which arises in any coordination game with Pareto-dominated equilibria. While expectations do play a role in the traditional fragility literature (e.g., Fisher 1933, Keynes 1936, and Minsky 1977), crises are also driven in that literature by economic fundamentals. This paper looks only at crises initiated by changes in fundamentals: actual project failures and loan defaults fuel actual-default crises, and the anticipation of these events generates anticipated-default crises. Neither type of crisis arises from investors reallocating their portfolios only because they think others will do so, although collectively they would be better off not reallocating. Since coordination-failure equilibria are not particularly relevant for understanding fragility based on fundamentals, the remainder of the paper works with a refinement of the symmetric equilibrium defined above that excludes equilibria resulting from sunspots or other coordination failures.

Additionally, since a goal of this paper is to assess whether economies are inherently fragile, use of the refinement of equilibrium is an analytical device that ensures that the equilibrium studied is the one that keeps as much of the chain structure intact for as long as possible, thus minimizing the likelihood of the economy's being labeled "fragile." To accomplish this, the refinement embodies optimistic forecasts of strategies, where an optimistic forecast is defined as follows.

DEFINITION. An *optimistic forecast* is a forecast $\tilde{f}^{-i}$ such that, for all $t \geq 0$ and all $j \neq i$, $\tilde{f}_t^{\,j}(W_t, k) = D$ and $\tilde{f}_t^{\,j}(H_t, k) \neq D$.

An investor with this forecast expects other investors to remain diversified if they have not yet incurred a loss, and to shift to safer portfolios after incurring a loss. He is thus as optimistic as possible about the future evolution of the chain structure, while still foreseeing the possibility of an actual-default crisis.[14] The refined equilibrium concept states that an investor who has not yet incurred a loss (one in state W_t) and who makes such a forecast chooses to shift to a safer portfolio, even if he expects other diversified investors to maintain their portfolios, if and only if the expected lifetime utility from deviating to the next-best alternative portfolio exceeds that from remaining diversified. Otherwise, the investor maintains his portfolio. There are, in general, different ways of achieving each alternative type of portfolio. An investor in state W_t compares the expected lifetime utility at the end of date t from his continuing to hold a diversified portfolio at $t + 1$, denoted $V_t^D(k)$ for simplicity, to the expected lifetime utility from deviating to the best portfolio of each alternative type. These expected utilities, given state W_t, are denoted $V_t^U(k)$, $V_t^P(k)$, and V^S for the undiversified, the part-safe-part-risky, and the safe alternatives, respectively.[15] The refined equilibrium, which is appropriately called a *maximal sustainable equilibrium*, is formalized in the following definition:

[14] Although investors with an optimistic forecast foresee the possibility of contagious defaults spreading to them, they do not foresee the possibility of all investors switching simultaneously to safer portfolios, setting off an anticipated-default crisis. Thus, investors who make the optimistic forecast are more optimistic than they would be with perfect foresight.

[15] That is, $V_t^a(k) \equiv V_t\left(f^i, \tilde{f}^{-i} | W_t, k\right)$ if $f_t^i = a$. However, the post-return lifetime utility from a safe portfolio, one with no funds invested in risky projects, is independent of k and constant for all $t \geq 0$.

DEFINITION. A *maximal sustainable equilibrium* is a symmetric equilibrium such that, for each date $t \geq 0$ and state *k*, each investor in state W_t uses strategy $f_t^{i*}(W_t, k) = D$, given his optimistic forecast, iff $V_t^D(k) \geq \max\{V_t^U(k), V_t^P(k), V^S\}$.

It is indeed an equilibrium because (1) all investors are identical ex ante, having identical utility functions over wealth consumed, and (2) it is a best response for each diversified investor to remain fully diversified given that he expects others like himself to do the same. Moreover, it is the unique Pareto-efficient equilibrium in the class of symmetric equilibria studied.

C. Expected Utilities

Because Property 2 pins down the equilibrium behavior at $t \geq 0$ of an investor in state H_t and investors in state Z_t have no funds to invest, solving for an equilibrium requires calculating and comparing the expected lifetime utilities of the various portfolios only for an investor in state W_t (that is, $V_t^D(k), V_t^U(k), V_t^P(k), V^S$). The expected utility of every portfolio with contagion risk has an upper and a lower bound that is constant and independent of contagion risk. The upper bound is associated with a zero probability of loss, while the lower bound is associated with a probability of loss of one. There is no need to calculate the actual expected utility when the upper bound is less than $V_t^D(k)$ for all *t*.

To calculate the actual expected lifetime utility from a portfolio with contagion risk, an investor must consider the probability of his being in each possible open or closed chain, given his knowledge of the states *k* and s_t (= W_t) and the date, which indicates how much time has passed since shocks were realized and thus how open chains could have evolved. The thought process used to derive the actual expected utilities is the same for each portfolio type. Consequently, the remainder of this subsection describes that process only for $V_t^D(k)$, the expected utility if the investor continues to hold a diversified loan portfolio. The actual expected utilities from deviating to a different type of portfolio are constructed in Appendix A, along with the expected utilities in states Z_t and H_t.

The expected utility from being in an open chain takes into consideration the utility associated with the various positions in the chain and the likelihood of an investor's being in each position. Figure 5 below illustrates the anatomy of an open chain at the beginning of any date $t \geq 1$.

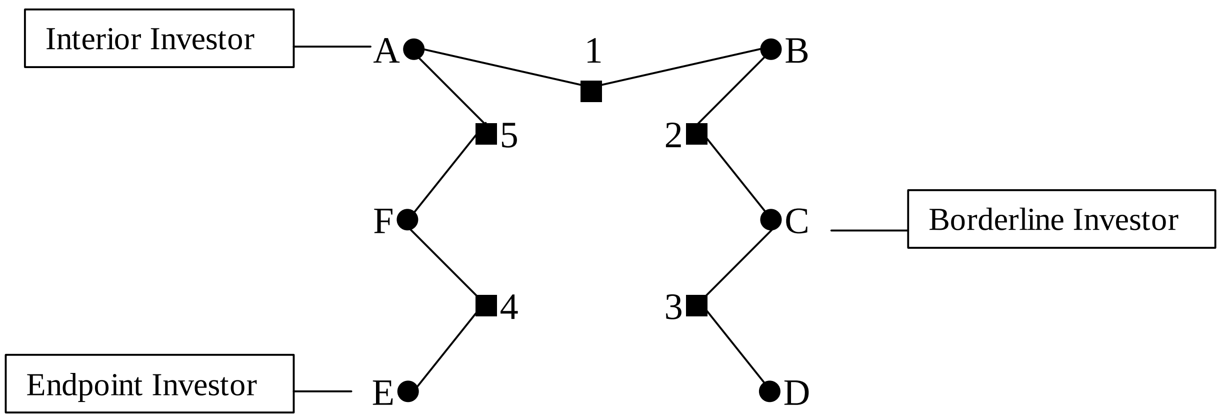

Figure 5—Some Investor Types

An open r-link chain has $r + 1$ investors. Two of them are on the endpoints of the chain. These "endpoint investors" have already lost \$1 from the default of one loan, so Property 2 describes their equilibrium behavior. That leaves $r - 1$ investors in the chain who are in state W_t. Two of these investors are "borderline investors" in that they hold portfolios with one loan to a project that will end up fully funded and one to a project that will end up insufficiently funded. They thus will receive a gross return of $\overline{R}$, and by Property 2, they will either consume it immediately or consume their interest and reinvest their principal in the safe asset for one period. Their utility is $\max\{u(\overline{R}), u(\overline{R} - 1) + bu(1)\}$. The probability of being in state W_t and a borderline investor is $2/(r - 1)$. It follows that the probability of being in state W_t and an "interior investor," one far from the ends of an open chain, is $1 - 2/(r - 1)$. The interior investor holds a portfolio of loans to two projects that will end up fully funded and thus will receive the return $2\overline{R}$. If he is to remain diversified, which is the case being considered, then he must consume his interest, which yields a current-period utility of $u(2\overline{R} - 2)$, and reinvest his principal. Because the chain's linkages are known to break over time, the expected utility at t from reinvesting his principal is the discounted expected utility from his being in an open $(r - 2)$-link chain at t+1. Ultimately the investor will be in the situation of a borderline investor.

In summary, the expected utility from remaining diversified if in an open r-link chain, which is denoted $p^D(r)$, may be defined recursively by

$$p^D(r) = \begin{cases} \frac{2}{r-1}\max\left\{u\left(\overline{R}\right), u\left(\overline{R}-1\right)+bu(1)\right\} + \left(1-\left(\frac{2}{r-1}\right)\right)\left(u\left(2\overline{R}-2\right)+bp(r-2)\right) & \text{if } r \geq 3, \\ 0 & \text{if } r = 2. \end{cases}$$

When $r = 2$, the investor is borderline with respect to both investments, so $p^D(2) = 0$. And $r = 1$ is not a relevant case because it requires the investor to be in state H_t, not W_t.

In contrast, investors in closed chains have not had any shocks hit projects in their chain. Thus, they can remain diversified permanently, enjoying the lifetime utility $\overline{p} \equiv u\left(2\overline{R}-2\right)/(1-b)$. The utility $\overline{p}$ also is precisely the limit of the expected utility from remaining diversified in an open r-link chain as r approaches infinity: $\overline{p} = \lim_{r\to\infty} p(r)$. The reason is that the larger an open chain is, the longer an investor can stay on the interior. The interior investor in an infinitely large open chain thus receives the same expected utility as a closed-chain investor.

In addition to these expected utilities from closed and open chains, the expected lifetime utility from remaining diversified, $V_t^D(k)$, depends on the probabilities of closed and open chains. With C_{rt} $\left(N_{rt}\right)$ denoting the event that an investor is in a closed (open) r-link chain at the beginning of date t, the probability of being in a closed (open) chain at the beginning of date t, conditional on his having entered the period in state W when there were k projects initially, can be denoted $P\left(C_{rt}|W_{t-1}, k\right)$ $\left(P\left(N_{rt}|W_{t-1}, k\right)\right)$. These probabilities would be the same if calculated after returns are realized at date $t-1$, when the investor makes his consumption-investment decision. It remains to derive $P\left(C_{rt}|W_{t-1}, k\right)$, $P\left(N_{rt}|W_{t-1}, k\right)$, and finally $V_t^D(k)$ for each $t \geq 0$.

Any closed r-link chain that exists after returns are realized at $t \geq 0$ had to have evolved from a closed r-link chain that existed at the start of date 0 and had no project failures due to shocks. The probability of an investor belonging to a closed r-link chain before the shocks hit at date 0 may be denoted $q(r,k)$. Clearly, $P\left(C_{r0}|W_{-1}, k\right) = q(r,k)$. Thus, at the start of any $t \geq 1$, the *nonnormalized* conditional probability of a closed r-link chain, denoted $\overline{P}\left(C_{rt}|W_{t-1}, k\right)$, is $q(r,k)(1-p)^{r-2}$ if $2 \leq r \leq k$ and zero otherwise (normalized probabilities are constructed below). The reason is twofold. First, by Property 1, there are no one-link chains at the start of date 0. Second, $(1-p)^{r-2}$ is the probability that none of the $r-2$ projects in the chain other than those in the investor's portfolio, which are known to have survived, were hit by shocks at date 0. The probability $\overline{P}\left(C_{rt}|W_{t-1}, k\right)$ is independent of t.

It remains to consider the probabilities of open chains. At the start of date 0, before any shocks have been realized, the nonnormalized probability of an investor's being in an open *r*-link chain, denoted $\overline{P}\left(N_{r0}|W_{-1},k\right)$, is zero. However, at the beginning of date 1, an investor can find himself in an open *r*-link chain. There are two routes by which this could happen. One possibility is that the open *r*-link chain came from a closed *m*-link chain at the start of date 0, with $m = r + 1$. That is, a single shock could have hit a closed *m*-link chain at date 0, creating an open chain. The probability of a closed *m*-link chain at the start of date 0 is $q(m,k)$, and there are $r - 1$ ways that *r* adjacent projects, two of which are in the investor's portfolio, can be chosen from the initial $r + 1$ projects. Thus, at the start of date 1, the probability of being in an open *r*-link chain that came from an initial closed *m*-link chain, with $m = r + 1$, is $(r - 1)q(m,k)p(1 - p)^{r-2}$.

Alternatively, the open *r*-link chain at the start of date 1 could have come from a closed *m*-link chain, $m > r + 1$, that existed at the start of date 0. In this case, at least two distinct projects in the original closed chain must have been hit by shocks and failed at date 0, but only the two projects whose failures created the ends of the open chain are relevant for the chain's evolution. Thus the probability of being in an open *r*-link chain that arose in this manner is $\sum_{m=r+2}^{k}(r-1)q(m,k)\,p^2(1-p)^{r-2}$.

It does not matter by which of these routes an investor finds himself in an open *r*-link chain at the start of date 1 because the same expected utility is associated with each case. The probabilities of the subcases can thus be summed to get $\overline{P}\left(N_{r1}|W_0,k\right)$, the nonnormalized total conditional probability that at the start of date 1 an investor resides in an open *r*-link chain, given states *k* and W_0:

$$\overline{P}\left(N_{r1}|W_0,k\right) = (r-1)(1-p)^{r-2}\left[p^2\sum_{m=r+2}^{k}q(m,k) + pq(r+1,k)\right].$$

In calculating the probability of open chains at later dates, the possibility of contagious defaults also must be considered. Because of contagion, chains at the start of dates $t > 1$ correspond one-to-one with certain open chains at date $t = 1$. For this reason, it is useful to first calculate $\overline{P}\left(N_{n1}|W_0,k\right)$, the normalized conditional probability that a diversified investor started date 1 in an open *n*-link chain. Clearly, $\overline{P}\left(N_{n1}|W_0,k\right) = 0$ if $n \geq k$ since at date 1 there could have been no open chains with *k* or more links when there were only *k* projects at the start of date 0. Likewise, $\overline{P}\left(N_{n1}|W_0,k\right) = 0$ if $n < 2 + 2(t - 1)$ because at the start of $t = 1$ any chain the investor could have resided in must have been large enough to contain the two projects in the investor's portfolio and all projects that would fail because of contagion between dates 1 and t (i.e., two per date for each date up through $t-1$). For all other open n-link chains at date 1 (those of sizes $2 + 2(t-1) \le n \le k-1$), $\overline{P}(N_{n1}|W_0,k)$ is positive and equal to $(n-1)(1-p)^{n-2}\left[p^2 \sum_{m=n+2}^{k} q(m,k) + pq(n+1,k)\right]$. It follows that $\overline{P}(N_{rt}|W_{t-1},k)$, the nonnormalized conditional probability of an investor being in an open r-link chain, satisfies $\overline{P}(N_{rt}|W_{t-1},k) = \overline{P}(N_{n1}|W_{t-1},k)$, where $n = r + 2(t-1)$.

Since each investor holding risky assets must be in either an open or a closed chain, the possibilities considered above are exhaustive. They can be normalized to sum to one by dividing them by $\Lambda_t(k)$, the total probability weight attached to all r-link chains, whether open or closed, in state k at date t:

$$\begin{aligned}\Lambda_t(k) &\equiv \sum_{r=2}^{k} \overline{P}(C_{rt}|W_{t-1},k) + \sum_{r=2}^{k} \overline{P}(N_{rt}|W_{t-1},k)\\ &= \sum_{r=2}^{k} \overline{P}(C_{rt}|W_{t-1},k) + \sum_{r=2+2(t-1)}^{k-1} \overline{P}(N_{r1}|W_{t-1},k)\\ &= \sum_{r=2}^{k} \overline{P}(C_{rt}|W_{t-1},k) + \sum_{r=2t}^{k} \overline{P}(N_{r1}|W_0,k).\end{aligned}$$

It follows that the probabilities of belonging to closed and open r-link chains, respectively, are

$$P(C_{rt}|W_{t-1},k) = \begin{cases} \overline{P}(C_{rt}|W_{t-1},k)/\Lambda_t(k) & \text{if } 2 \le r \le k \\ 0 & \text{otherwise} \end{cases}$$

and

$$P(N_{rt}|W_{t-1},k) = P(N_{n1}|W_{t-1},k) = \begin{cases} \overline{P}(N_{n1}|W_0,k)/\Lambda_t(k) & \text{if } 2t \le n \le k-1 \text{ and } n = r + 2(t-1) \\ 0 & \text{otherwise.} \end{cases}$$

Using these normalized probabilities, the post-return expected lifetime utility at t for an investor in state W_t who makes an optimistic forecast and chooses to remain diversified, $V_t^D(k)$, can be written as[16]

$$\begin{aligned}V_t^D(k) \equiv V_t\left(f^i, \tilde{f}^{-i}|W_t,k\right) = u(2\overline{R}-2) + b\Bigg\{&\sum_{r=3}^{k}\left[P(N_{r\,t+1}|W_t,k)\left(\frac{2}{r-1}\right)V_{t+1}\left(f^i,\tilde{f}^i|H_{t+1},k\right)\right]\\ &+\left[\sum_{r=3}^{k} P(N_{r\,t+1}|W_t,k)\left(1-\frac{2}{r-1}\right) + \sum_{r=2}^{k} P(C_{r\,t+1}|W_t,k)\right]V_{t+1}\left(f^i,\tilde{f}^{-i}|W_{t+1},k\right)\\ &+ P(N_{2t+1}|W_t,k)V_{t+1}\left(f^i,\tilde{f}^{-i}|Z_{t+1},k\right)\Bigg\}, \text{ for } t \ge 0 \text{ and } f_t^i(W_t,k) = D.\end{aligned}$$

[16] The investor's problem is recursive from date 0 on, not from date –1 on. The date –1 expected lifetime utilities are derived in Appendix B.

Clearly, $V_t\left(f^i, \tilde{f}^{-i} | Z_t, k\right) = 0$ for all $t \geq 0$, so $V_t^D(k)$ can be rewritten as

$$V_t^D(k) \equiv V_t\left(f^i, \tilde{f}^{-i} | W_t, k\right) = u\left(2\bar{R} - 2\right) + b\left\{\sum_{r=3}^{k}\left[P\left(N_{r\,t+1} | W_t, k\right)\left(\frac{2}{r-1}\right)V_{t+1}\left(f^i, \tilde{f}^{-i} | H_{t+1}, k\right)\right]\right.$$
$$\left. + \left[\sum_{r=3}^{k} P\left(N_{r\,t+1} | W_t, k\right)\left(1 - \frac{2}{r-1}\right) + \sum_{r=2}^{k} P\left(C_{r\,t+1} | W_t, k\right)\right] V_{t+1}\left(f^i, \tilde{f}^{-i} | W_{t+1}, k\right)\right\},$$

for $t \geq 0$ and $f_t^i(W_t, k) = D$.

By the law of iterated expectations,

$$E_t\left[P\left(C_{r\,t+1+h} | W_{t+h}, k\right) | W_t, k\right] = P\left(C_{r\,t+1+h} | W_t, k\right)$$

for all $h > 0$, and

$$P\left(C_{r\,t+1+h} | W_t, k\right) = \frac{q(r,k)(1-p)^{r-2}}{\Lambda_t(k)} = P\left(C_{r\,t+1} | W_t, k\right).$$

Likewise,

$$E_t\left[P\left(N_{r\,t+1+h} | W_{t+h}, k\right) | W_t, k\right] = P\left(N_{r\,t+1+h} | W_t, k\right)$$

and

$$P\left(N_{r\,t+1+h} | W_t, k\right) = P\left(N_{n\,t+1} | W_t, k\right).$$

for $n = r + 2(h - 1)$. Solving the expression for $V_t^D(k)$ forward thus yields the simpler expression

$$V_t^D(k) = u\left(2\bar{R} - 2\right) + b\left\{\sum_{r=2}^{k}\left[P\left(C_{r\,t+1} | W_t, k\right)\bar{p} + P\left(N_{r\,t+1} | W_t, k\right)p^D(r)\right]\right\}, \; t \geq 0.$$

This completes the derivation of $V_t^D(k)$.

D. Characterizing Ex Ante Fragility

To characterize ex ante fragility, the date at which an anticipated-default crisis occurs must be determined. It is clear that once the shocks hit at the initial date, some closed chains become open chains. By Property 2, in an equilibrium open chains shrink in size over time. Without information on how many chains are left, investors still holding diversified portfolios may view the risk of defaults spreading to them as increasing, at least over some period of time, because the probability of being in smaller open chains is increasing.[17] Thus, it is reasonable to expect there to be a first date at which investors choose to shift to a safer portfolio. This idea is formalized in the

[17] At some point, they will view the risk of contagious defaults as decreasing because they know they are almost certainly in a closed chain.

following proposition:

PROPOSITION 1. Let f^* be any maximal sustainable equilibrium. There exists an exit-decision date $t(k)$ for each k that satisfies for all i,

$$f_t^{i*}(W_t,k) = D \quad \text{if} \quad t < t(k), \text{ and}$$
$$f_t^{i*}(W_t,k) \neq D \quad \text{if} \quad t \geq t(k).$$

That is, a date $t(k)$ exists that is the earliest date at which all investors in state W_t consider the risk of impending defaults to be so great that $V_t^D(k) < \max\{V_t^U(k), V_t^P(k), V^S\}$, so they decide to shift to a safer portfolio at the next date. These portfolio reallocations shatter the chain structure. By the construction of a maximal sustainable equilibrium, $t(k)$ is the exit-decision date that keeps the chain structure together as long as possible. This proposition suggests the following result regarding the characterization of fragility:

COROLLARY. An economy is *increasingly fragile* the smaller is $t(k)$. In particular, if $t(k) = 0$, the economy suffers an anticipated-default crisis at date 1, while if $t(k) = \infty$, then investors never preemptively switch to safer portfolios and thus remain diversified until they experience a loss.

For the economy's financial structure to remain intact for some intermediate amount of time (i.e., for $0 < t(k) < \infty$), it is necessary that $V_0^D(k) > \max\{V_0^U(k), V_0^P(k), V^S\}$ and that $V_0^D(k)$ strictly decreases over some range of dates $t = 0,\ldots,t'$. If Property 1 holds, so that investors prefer to hold diversified portfolios before any shocks are realized, then they most likely prefer to hold diversified portfolios immediately after the shocks hit. But it need not be the case that $V_t^D(k)$, or more generally the expected lifetime utility from holding any portfolio with contagion risk, is strictly decreasing for small t. The reason depends both on the probabilities of the chains and on the expected utilities of various chains.

At each date, the probability of investors being in the largest open chain that existed at the previous date goes to zero. In particular, an open $(k-1)$-link chain is possible at $t = 1$ but not at $t = 2$. More generally, since the largest open chain at the beginning of $t = 1$ is of size $k - 1$, open r-link chains at dates $t \geq 1$can have no more than $k - 2t + 1$ links. Hence, the probability mass assigned to

$p^D(k-1)$ at t is reassigned to the expected utility of other chains at $t + 1$. There are two countervailing effects. First, some of the probability mass is assigned to $\bar{p}$, the utility from being in a closed chain. That is, as time passes, it becomes more likely that an investor who has not incurred a loss belongs to a closed chain. In fact, as the discussion above showed, no contagion can continue beyond $t = (k - 1)/2$, so investors must be in closed chains at all later dates. This is the best possible scenario from the investor's perspective. Second, some of the mass is reassigned to the expected utility from smaller open chains. This is bad for the investor because contagious defaults reach him sooner in smaller open chains. Thus, for the expected utility from a portfolio with contagion risk to decrease initially (e.g., for $V_0^D(k) > V_1^D(k)$), relatively less probability weight must be reassigned to $\bar{p}$ than to the expected utilities $p^D(r)$ for small r, and $\bar{p}$ must be sufficiently large relative to $p^D(r)$.

The movement of the expected utilities of chains is itself complicated, however. The utility from being in a closed chain is constant over time and the same as the utility from an infinitely large open chain. But the expected utility from an open chain is not monotonically increasing in chain size.[18] This is because an investor's position in an open chain has a large impact on his expected utility, and in small chains an investor can remain on the interior for fewer periods. In particular, an investor in an open three-link chain is necessarily a borderline investor. If he waits for defaults to spread to him, he loses one of his investments in the current period, which will lead him to not reinvest in risky assets at the next date. In contrast, an investor in an open four-link chain has a 2/3 chance of being a borderline investor and a 1/3 chance of being an interior investor. As an interior investor, if he waits for contagious defaults to affect him, he does not incur a loss in the current period. But he will find himself as the sole—and necessarily endpoint—investor in an open two-link chain at the subsequent date, which means that he will lose his entire principal. The expected utility from an open four-link chain ($p^D(4)$) therefore is less than that from an open three-link chain ($p^D(3)$). Since the expected utilities of larger chains with an odd number of links all depend on $p^D(3)$, and those with an even number of links depend on $p^D(4)$, $p^D(r)$ exhibits cycles for small values of r. The expected utilities from chains with an even number of links, however, do

[18] Since open chains shrink in size over time, if the expected utility from an open chain was monotonically increasing in chain size, then at least over some intermediate period the investor's expected utility from remaining diversified might be decreasing. Ultimately, the expected utility from remaining diversified would have to increase, however, since it must converge to the expected utility from being in a closed chain.

increase monotonically with chain size (i.e., $p^D(r) < p^D(r+2)$ for these chains) if b is sufficiently close to one. The same is true for the expected utilities from chains with an odd number of links. And for sufficiently large open chains, the weight on $p^D(3)$ or $p^D(4)$ is so small that the expected utility of an open chain is monotonically increasing in chain size. Figure 6, below, illustrates $p^D(r)$ for the preferences and parameter values that are shown in Appendix B to be consistent with an equilibrium. The cycles are apparent, and the function is strictly increasing for $r > 39$.

To make more concrete the calculation of chain probabilities, portfolio expected utilities, and the exit-decision date, subsection F below presents examples for various economies.

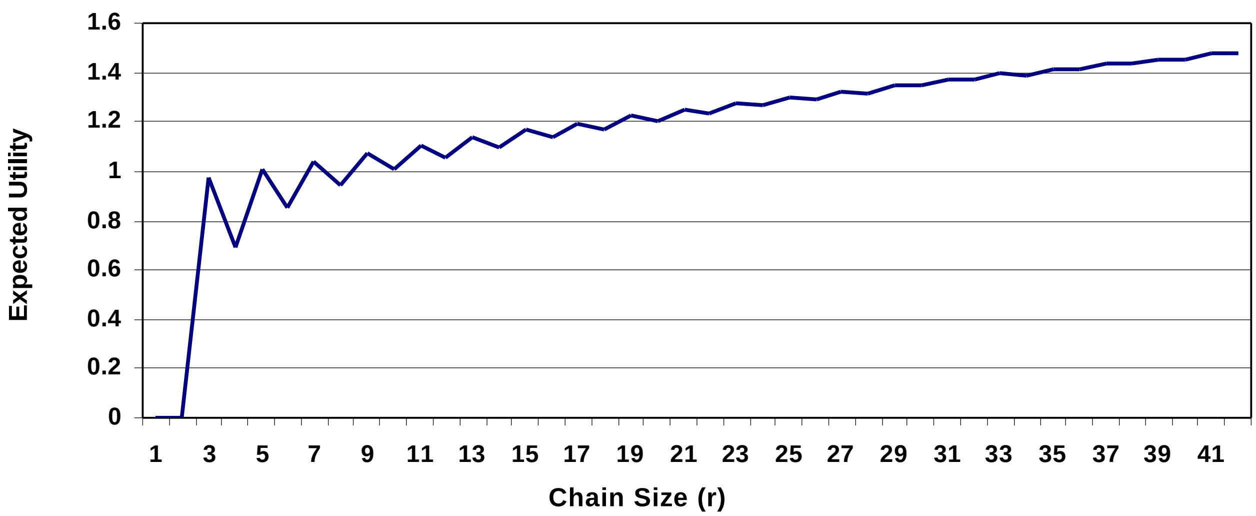

Figure 6—Expected Utilities from Open r-link Chains

E. Fragility and Economy Size

These results raise the question of how fragility depends on the size of the economy. It is reasonable to expect that as an economy gets larger (i.e., as the number of entrepreneurs, investors, and investment projects, as reflected by the parameter k, increases), holding fixed the degree of diversification, it also becomes more fragile. In other words, $t(k)$ should be decreasing in k. The next result shows that, for sufficiently large economies, financial fragility indeed worsens as the economy increases in size.

PROPOSITION 2. Let $q(r,k)$ denote any prior over closed chains that satisfies $\lim_{k\to\infty} q(ak,k) = 1$ for some a, where $0 < a < 1$. Then if b is sufficiently close to one, there exists a k' such that, $\forall\ k \geq k'$,

$t(k)$ is decreasing in k.

The proof appears in Appendix C. The intuition behind it relies on three observations. First, Proposition 2 requires that $q(r,k)$ place full mass on chains of a particular size relative to k as k approaches infinity. While this assumption is somewhat restrictive, it is consistent with most theories of random and unbiased agglomeration. The reason is that random and unbiased agglomeration requires that investors place mass on all possible combinations of other investors. But combinations are derived from the binomial coefficients $\binom{k}{r}$. In turn, the binomial coefficients place relatively increasing weight on the median combination $\binom{k}{k/2}$ as k increases. Hence, the chain size with the coefficient $\binom{k}{k/2}$ attached receives full mass in the limit as k approaches infinity.

Second, as $k \to \infty$, the total probability of being in a closed chain after the shocks hit at $t = 0$ approaches zero because the average number of shocks realized—kp—is increasing in k. Thus, the larger is k, the more likely it is that an investor is in an open chain after the shocks hit.

Third, the projects that are hit by shocks are as likely to be distributed around a closed chain one way as they are any other way. On average, then, realized shocks are uniformly distributed around a chain.

Combining these three observations yields the average size of an open chain after the shocks have hit at the initial date. All such open chains are formed from shocks hitting closed chains. The average number of shocks hitting the average-size initial chain—a closed αk-link chain—is αkp because there are $1/\alpha$ chains of average size, and a fraction α of the uniformly distributed shocks must hit α of the chains. The average size of the open chains after shocks hit is then $\alpha k/\alpha kp$, or $1/p$, which is independent of k. Thus, as k increases, the likelihood of being in an open chain at the start of $t = 1$ also increases, while the average size of an open chain stays the same. As a result, the risk of loss due to contagion increases in k. Since the exit-decision date $t(k)$ is the first date for which $V_t^D(k) < \max\{V_t^U(k), V_t^P(k), V^S\}$, $t(k)$ is decreasing in k if the difference between $V_t^D(k)$ and $\max\{V_t^U(k), V_t^P(k), V^S\}$ is decreasing in k. If $\max\{V_t^U(k), V_t^P(k), V^S\} = V^S$, for example, this simply means that $V_t^D(k)$ must be decreasing in k. Otherwise, $V_t^D(k)$ must be decreasing more than $\max\{V_t^U(k), V_t^P(k), V^S\}$. The proof of the proposition shows that this is indeed the case as k approaches infinity when $q(r,k)$ satisfies the condition stated. Thus, as economies become larger, with the degree of diversification given, they also become more fragile.

One prior satisfying the condition of Proposition 2 is, for fixed k,

$$q(r,k) = \frac{\binom{k-1}{r-1}}{\sum_{n=1}^{k-1}\binom{k-1}{n} - \binom{k-1}{k-2}}, \quad r = 2,\ldots,k-2,k\,. \tag{1}$$

Investors with this prior view each possible chain of each possible size as equally likely. Moreover, this prior embodies the fact that by Property 1, there are no initial closed r-link chains for $r = k-1$ or $r = 1$. Therefore, $q(k-1,k) = q(1,k) = q(0,k) = 0$. And naturally, $q(r,k) = 0$ if $r > k$.[19]

F. Examples

Some numerical examples for economies of various sizes nicely illustrate the results presented in this section. In the examples to follow, the prior $q(r,k)$ is from equation (1), and preferences and parameter values are the same as those used in Appendix B to show that Properties 1 and 2 are consistent with the equilibrium considered. Specifically, $u(c)=(c-a)^{1-s}$ for $c>a$ and $u(c)=0$ otherwise, $s = 0.20$, $a = 0.03$, $\bar{R} = 1.05$, $b = 0.95$, $p = 0.01$.

Examples for the $k = 4$ and $k = 10$ economies make clear how investors calculate $V_t^D(k)$. Table 1 summarizes the results for the $k = 4$ economy. Columns 3 and 4 of the table show the nonnormalized conditional probabilities of closed and open chains, while columns 5 through 8 show the corresponding normalized conditional probabilities. Open chains are possible at the start of date 1 (column 6), but not date 2 (column 8). No diversified investor can be in an open chain at the start of date 2 because the largest open chain that occurs with positive probability at date 1 has three links. By date 2, that chain will have lost enough links that the only person remaining in it will be an endpoint investor. $\Lambda_2(4)$ spreads the probability weight that had been assigned to open chains

[19] Alternatively, suppose that the investor views all chain structures as equally likely. He then assigns probability $\binom{k-1}{r-1}Q(k-r)\big/Q(k)$ to the event that he belongs to a closed r-link chain, where $Q(m)$ is the number of ways m objects can be partitioned, given that there are no partition elements of sizes 1 and $k-1$. One can verify that this prior satisfies the condition in Proposition 2.

at date 1 across closed chains at date 2.

r	$q(r,4)$	$\overline{P}(C_{rt} \vert W_t,4)$ for all t	$\overline{P}(N_{r1} \vert W_1,4)$	$P(C_{r1} \vert W_1,4)$	$P(N_{r1} \vert W_1,4)$	$P(C_{r2} \vert W_2,4)$	$\overline{P}(N_{r2} \vert W_2,4)$, $P(N_{r2} \vert W_2,4)$
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.75	0.75	0.000025	0.75	0.000025	0.75375	0.0
3	0.0	0.0	0.00495	0.0	0.00495	0.0	0.0
4	0.25	0.245025	0.0	0.245025	0.0	0.24625	0.0
				$\Lambda_1(4)$ =1.0		$\Lambda_2(4)$ =0.995025	

Table 1—Probabilities of Chains for the *k* = 4 Economy

The expected utility from remaining diversified in the *k* = 4 economy is shown in the second column of Table 2 below. It converges to the expected utility from being in a closed chain by date 2. That result corresponds to the zero probability of open chains (column 5) as of date 2.

Table 2 also compares the results for the *k* = 4 economy to those for the *k* = 10 economy. It shows that $V_t^D(10)$ falls and then rises and converges to 2.38, the expected utility of a closed chain. The reason is that the probability of being in a large open chain (column 6, with “large” arbitrarily defined to be more than five links) decreases by 0.012 from date 1 to date 2, while the total

t	$V_t^D(k)$ for k=4	$V_t^D(k)$ for k=10	Prob. Of Residing in Closed Chain, k=10	Prob. Of Residing in Open Chain, k=10	Prob. Open r-Link Chain, $r > 5$, k=10	Prob. Open r-link Chain, $r \le 5$, k=10
1	2.374	2.333	0.9661	0.0338	0.0122	0.0216
2	2.38	2.331	0.9703	0.0297	0.0002	0.0295
3	2.38	2.355	0.9875	0.0124	0.0	0.0124
4	2.38	2.38	1.0	0.0	0.0	0.0
5	2.38	2.38	1.0	0.0	0.0	0.0
6	2.38	2.38	1.0	0.0	0.0	0.0
7	2.38	2.38	1.0	0.0	0.0	0.0
8	2.38	2.38	1.0	0.0	0.0	0.0
9	2.38	2.38	1.0	0.0	0.0	0.0
10	2.38	2.38	1.0	0.0	0.0	0.0

Table 2—Analysis of *k* = 10 Economy and Comparison to *k* = 4 Economy

probability of being in an open chain (column 5) decreases only by 0.0041. The implication is that the probability weight formerly given to large open chains is spread between small open chains and closed chains. The probability of small open chains (column 7) increases by 0.0079, and the probability of closed chains (column 4) increases by 0.0042. Since the probability of small open chains rises more than the probability of closed chains from date 1 to date 2, and the expected utility from smaller open chains is lower than that from larger ones (illustrated for the same preferences and parameter values in Figure 6 above), the expected utility from remaining diversified decreases.

Figure 7 below shows the determination of the exit-decision date for an economy with k = 100. The expected lifetime utility from the best portfolio of each type is plotted. For portfolios involving contagion risk, the expected lifetime utility has a v shape. This is illustrated by $V_t^D(100)$ and $V_t^P(100)$, and is consistent with the example in Table 2. The initial downward slope arises because the conditional probability of small open chains initially increases more than the conditional probability of closed chains, and because the expected lifetime utilities of open chains are sufficiently lower than the utilities of closed chains. Eventually the expected utility function becomes upward sloping because the conditional probability of being in a closed chain dominates.

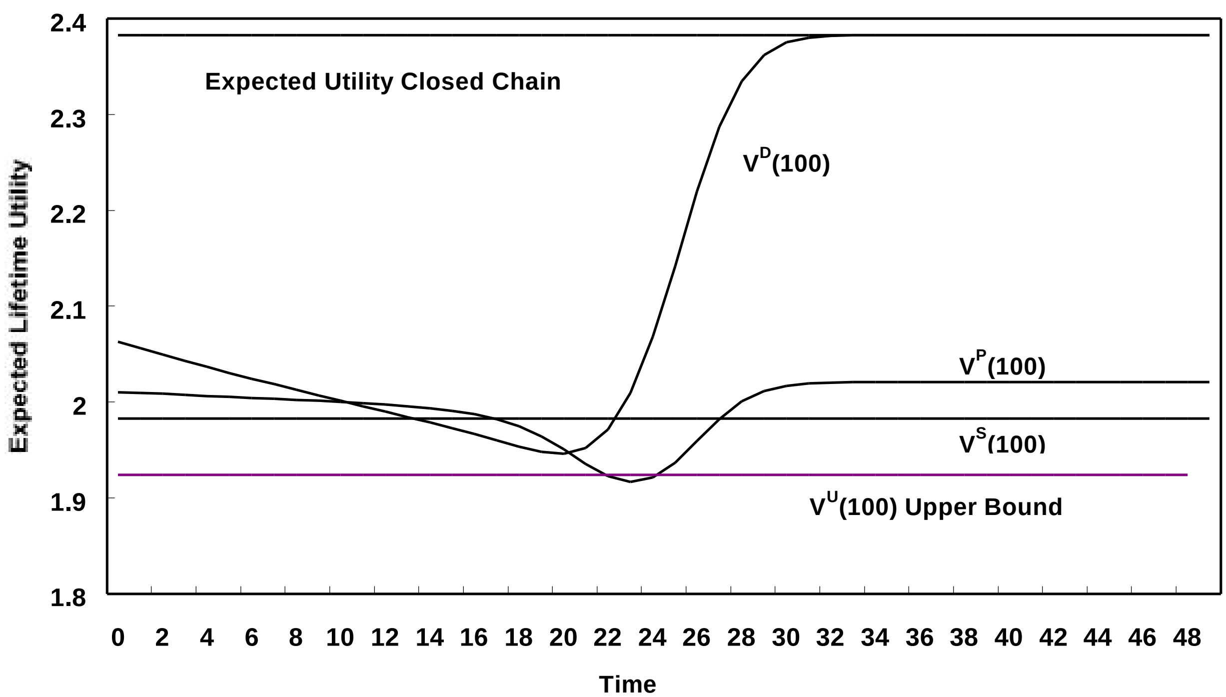

Figure 7—Determination of the Exit-Decision Date

It converges to its upper bound, the expected utility from holding the portfolio when in a closed chain, exactly when it should, at the start of date $(k-1)/2$, which here is date 49. This same reasoning is behind $V_t^P(k)$ being flatter than $V_t^D(k)$. Because the part-safe-part-risky portfolio involves fewer dollars at risk from contagion than the diversified portfolio, the effect of probability weight shifting primarily first to small open chains and later to closed chains is mitigated in the former portfolio.

Of the alternative portfolios, the part-safe-part-risky portfolio is the best in this example. The exit-decision date is thus the first date for which $V_t^D(100) < V_t^P(100)$. In this example, it is date 11. That is, all investors decide at date 11 to reinvest only \$1 in one risky investment, causing the chain structure to collapse at date 12, when they implement their decisions.

The safe portfolio is the next best alternative, followed by the undiversified portfolio. $V^S(100)$ is a constant, as it should be. The upper bound on $V_t^U(100)$ is less than the expected utility from the other portfolios, so it is all that is shown.

Figure 8 below illustrates how $t(k)$ changes with the size of the economy. The expected lifetime utilities for the diversified portfolio and the best alternative—the part-safe-part-risky portfolio—are shown for economies with k equal to 50, 100, and 150. For k = 50, $t(k) = \infty$. This is also the case for all smaller economies and for somewhat larger economies. This result is not surprising. It obtains partly because the expected lifetime utility from continuing to hold a diversified portfolio at any date is calculated conditional on the investor's knowing that he has not yet experienced a loss. In a small economy, an investor's portfolio contains loans to a large share of the economy's projects. And because contagion causes a minimum of two projects to fail per period, over time the investor's projects become a larger share of the total remaining. Thus, if the economy is relatively small, it does not take very long before an investor thinks he is most likely in a closed chain. The figure also shows that for sufficiently large economies, $t(k)$ is decreasing in k. For the example studied, if k = 100, t is 11, and if k increases to 150, t decreases to 2.

Figure 8 highlights a related point. Because the fragility properties of small economies differ from those of large economies, it is instructive, but misleading, to study only examples for small economies.

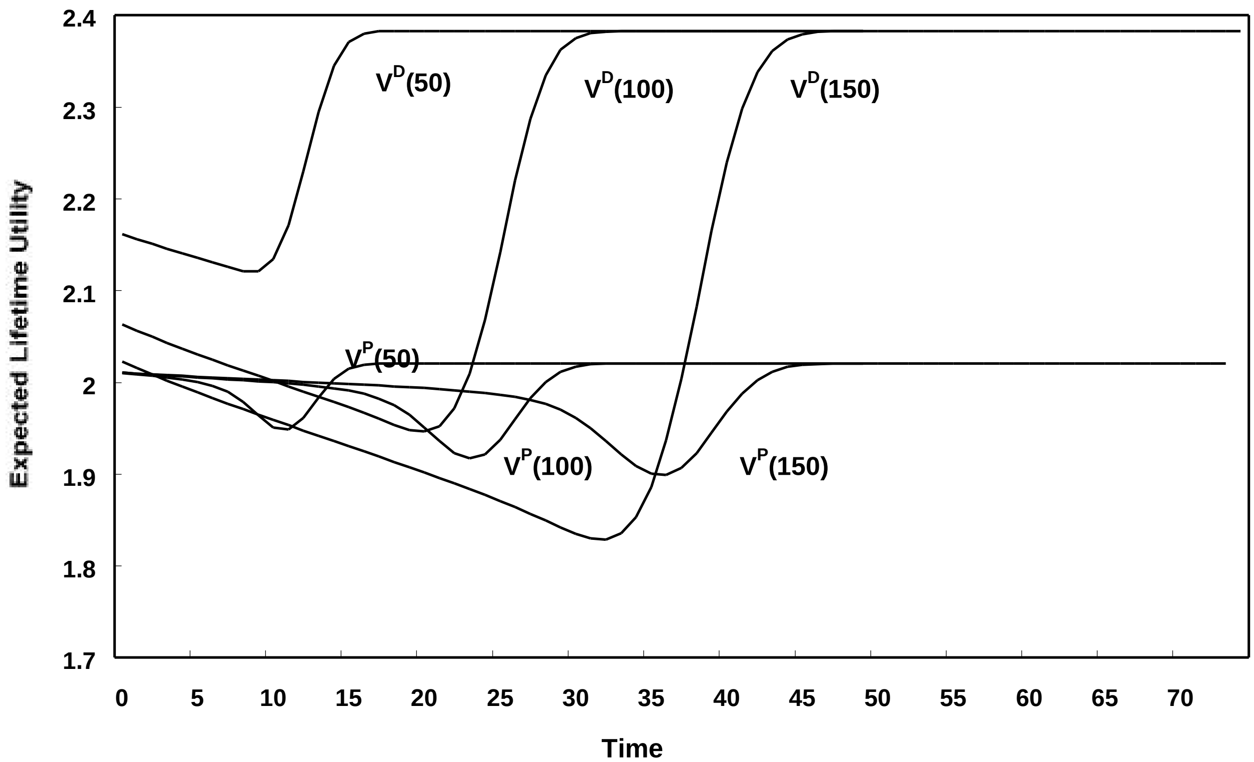

Figure 8—Fragility Increases with Economy Size

V. Diversification

Thus far this paper has taken the degree of diversification as given and examined the impact of greater financial interconnectedness on financial fragility. A more challenging task is to take the degree of interconnectedness as given and explain the impact of greater diversification on fragility. While studying the effect of diversification in general is quite difficult, insight can be gleaned from the following example, which is based on a modified version of the model considered thus far.

Greater diversification can be introduced most simply by assuming that each investor is endowed with three instead of two indivisible dollars and using modified versions of Properties 1 and 2:

PROPERTY 1 (MODIFIED). At $t = -1$, a maximum-return diversified loan portfolio, one with three \$1 loans to distinct entrepreneurs who end up fully funded, is preferred to any other feasible portfolio.

PROPERTY 2 (MODIFIED). At each $t \geq 0$, an investor who has incurred a loss of 50 percent or more of his portfolio prefers not to reinvest in risky assets.

Thus, investors are more diversified in this modified economy in that they hold 33 percent of their wealth in a loan to a particular project versus 50 percent in the economy studied above. The degree of financial linkage can be held fixed by using Properties 1 and 2, holding k constant, and assuming that portfolios are such that chains are still symmetric.

With these assumptions, the initial chain structure when $k = 4$, for example, is necessarily a single closed four-link chain. This structure could look as illustrated in Figure 9 below. Each investor is *directly* linked to the project on his left and the two closest projects on his right.

It is readily apparent, given the assumptions on preferences, that the single closed four-link chain structure is less subject to financial crisis when there is greater diversification because only a

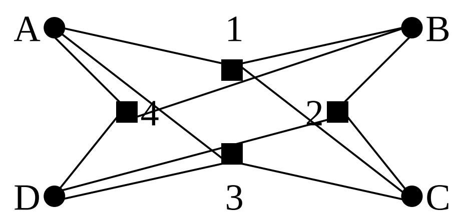

Figure 9—The Effect of Greater Diversification on a Closed Four-Link Chain

subset of the possible realizations of shocks can initiate an actual-default crisis.[20] With an actual-default crisis less likely, an investor's probability of being in an open chain after the initial shocks is lower, making an anticipated-default crisis less likely too. Specifically, if only one project in the economy, say project 4, is hit by a shock, then investors A, B, and D suffer portfolio losses. By Property 2, however, their losses are not large enough to induce them to pull out of the chain structure. Since no portfolio reallocations occur at the start of date 1 as a result of the shocks, no losses spread to investor C. And the chain remains closed, as illustrated in Figure 10, despite the failure of a project. Thus, for either type of financial crisis to occur in this economy with greater

[20] This assumes that projects are exactly fully funded with either \$2 or \$3 invested and insufficiently funded with less than \$2 invested in them.

diversification, at least one investor must at the initial date suffer losses on at least two of the loans he made. Greater diversification, then, holding fixed the linkage structure, economy size, and preference assumptions governing when an investor reallocates, makes an economy less fragile.

In general, as an economy increases in size, one would expect the degree of diversification and the degree of financial linkage to increase. Which dominates, the benefits of greater diversification or the costs of greater interconnectedness, remains the subject of future research.

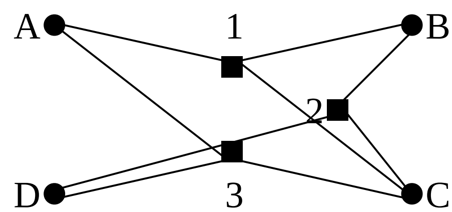

Figure 10—With Greater Diversification, a Closed Chain Can Remain Closed after a Shock Hits

VI. Institutional Responses to Fragility

Some comments on the model's policy implications are appropriate. The equilibria examined here are Pareto efficient in the class of symmetric equilibria with Properties 1 and 2, but financial crises do occur, so fragility is costly. This model is very primitive, abstracting from any institutions or government policy that might help overcome crises. For example, mutual funds and financial intermediaries are obvious candidates for financial institutions that would naturally arise to enhance risk sharing in this economy. These institutions can be introduced into the model simply by reinterpreting the individual investors as mutual funds. Likewise, the entrepreneurs could be reinterpreted as sectors. None of the model's results, however, would be affected by these changes. Unless all sectors and all investors share risk, financial crises are still possible, and the economy is still potentially fragile.

Alternatively, the government could intervene to reduce fragility. In fact, as the opening quotation from Friedman illustrates, some economists argue that financial crises represent a failure of government policy. As Schwartz (1986, p. 12) puts it, "A real financial crisis occurs only when institutions do not exist, when authorities are unschooled in the practices that preclude such a development, and when the private sector has reason to doubt the dependability of preventive arrangements." One often-prescribed intervention is for the government to function as a lender of last resort to prevent financial crises. If the government controls the supply of objects known as dollars in this model, then it can serve as a lender of last resort, lending to entrepreneurs who otherwise would be insufficiently funded. The government can serve this role even if it has the same information as private agents about the chain structure and spreading of defaults; all that is necessary is for the government to be known to all agents and able to broadcast announcements. Given such capabilities, if the government announces after returns are realized at date 0 that it will immediately begin serving as a lender of last resort, it can bring about a Pareto-optimal allocation.[21] The reason is that there are no incentive-compatibility problems here: entrepreneurs have no incentive to misrepresent themselves to obtain extra funds to finance additional or riskier investments. In a more general model, however, such incentive problems would arise, and it is not clear that it would be optimal for the government to serve as a lender of last resort.

Another possibility is for the government to sell insurance against investment losses from contagious defaults. Investors could buy such insurance after returns are realized at dates $t \geq 0$. The funds raised from insurance sales would keep projects with insufficient private-sector funding in operation. Again, there would be no incentive-compatibility problems with providing such insurance because of the simple structure of the entrepreneur's problem. How high the premium would be will depend on the solution to the investor's consumption problem, which in turn will depend on the degree of risk aversion. Economies with an insurance equilibrium should be less fragile than those without because the insurance essentially allows complete diversification: it connects each investor to the government, and through the government to all other investors in the economy. The insurance, of course, is just a type of tax-transfer scheme that brings about a Pareto-superior outcome. Solving for an insurance equilibrium remains the subject of future research.

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[21] The role for a government lender of last resort could be viewed as an artifact of the absence of outside investors from the model. But that view presumes that there is a financial system outside the model from which new investors could come to take the place of those who have experienced losses. That is, it presumes that the model economy's financial linkages are limited by some boundaries (e.g., national boundaries). There is no reason for that to be the case. In fact, if the model is to reflect the financial fragility of a country's economy, then it must reflect all the financial linkages of that country's residents. In a world with global financial markets, it is more appropriate to interpret the model as that of a global economy, and the lender of last resort as some international entity that makes up for the absence of any outside investors and the market incompleteness caused by the assumptions of limited information and communication.

VII. Conclusion

This paper presents a model in which agents' financial positions are linked through the diversified portfolios they hold and the payment commitments that emerge from credit market activity. Shocks to the economy cause some entrepreneurs to default on their payment commitments and thus some investors to suffer losses on their portfolios. Because of the portfolio linkages, defaults can spread through the financial system, allowing the shocks to have an impact far beyond their place of origin. If investors do not foresee defaults spreading to them, then a financial crisis can occur in which the contagion runs its course, destroying some linkages. This equilibrium suggests characterizing fragility in terms of the severity of the resulting crisis. This approach seems natural and is the one typically used, but since the severity of crises is randomly determined in the model, the approach cannot yield definitive conclusions about fragility. If, instead, investors have foresight about contagion, then a financial crisis can occur in which all investors seek safer portfolios as protection against the mere possibility of future defaults. This equilibrium suggests characterizing fragility in terms of the speed with which the entire financial structure collapses. The sooner the collapse occurs, the more fragile is the economy. The model yields a date at which such a collapse unambiguously occurs. That date depends on the features of the environment that jointly determine which portfolios investors prefer. For sufficiently large economies, the date at which total financial collapse occurs decreases as the economies increase further in size, with the degree of diversification fixed. In contrast, greater diversification, holding fixed the degree of financial linkage, can reduce fragility. Various institutional responses to fragility are considered. In particular, if the government can control the supply of dollars, it can overcome fragility by serving as a lender of last resort because no incentive-compatibility problems exist.

The model is extremely simple and makes some highly specialized assumptions to generate minimally diversified and linked portfolios while keeping the investor's problem manageable. If the only concern was with portfolio allocations, these assumptions would indeed be quite restrictive. But here the concern is with fragility that arises from financial linkages and the propagation mechanism. Even with the minimally diversified and linked portfolios generated, myriad financial structures are possible, which makes analyzing the economy quite complicated. In a way, then, the model might not be restrictive enough.

Finally, several specialized features of the model deserve comment. One such feature is the assumption that shocks occur only at a single date. A related specialized feature is the assumption that projects can die but not regenerate. If new projects could enter the economy over time, investment cycles would arise. For the model to characterize such investment cycles, however, it would have to explain why one financial, or chain, structure forms instead of another. The model is silent on this issue. This omission is of some consequence, for it rules out feedback between the economy and the chain structure.[22] An endogenous chain structure could be introduced into the model here by allowing investors to update their priors over the possible chain structures based on the realization of financial crises over time. After an extended period without a crisis, for example, investors might begin to put more probability weight on the more resilient chain structures. This, in turn, could encourage them to choose more risky portfolios, thereby forging more links to other investors and making the financial structure more fragile. This process could continue until a crisis occurs and the chain structure collapses.

Appendix A

This appendix constructs the expected utility from each feasible portfolio action (*D*, *U*, *P*, or *S*) in each state (*W*, *H*, or *Z*). It assumes that the investor makes an optimistic forecast.

Clearly, an investor with no wealth can consume nothing, which, by assumption, yields zero utility:

$$V_t\left(f^{\,i}, \tilde{f}^{\,-i} | Z_t, k\right) = u(0) = 0\,. \tag{2}$$

An investor who finds himself in state H_t after returns are realized at any date $t \geq 0$ has wealth of $\overline{R}$ that he can put into either a safe portfolio or a part-safe-part-risky portfolio for an expected utility of

[22] The idea that there is feedback between the financial structure and the business cycle that makes the economy inherently unstable is laid out in the financial-instability hypothesis, articulated frequently by Minsky (e.g., 1975).

$$V_t\left(f^i,\tilde{f}^{-i}|H_t,k\right)=\begin{cases}\max\left\{u\left(\overline{R}\right),u\left(\overline{R}-1\right)+bu(1)\right\} \text{ if } f_t^i\left(H_t,k\right)=S,\\ u\left(\overline{R}-1\right)+b\left\{\left[1-P\left(N_{1\,t+1}|H_t,k\right)\right]V_{t+1}\left(f^i,\tilde{f}^{-i}|H_{t+1},k\right)\right\}\\ \quad \text{if } f_t^i\left(H_t,k\right)=P\,.\end{cases} \tag{3}$$

If he chooses a safe portfolio, he either consumes his entire wealth or consumes his interest and invests his remaining dollar in the safe asset. Either way, he leaves no funds at risk in the chain structure. If he holds a part-safe-part-risky portfolio, he must reinvest his remaining dollar in a risky asset (the project he held previously that has not yet failed) and pull the rest $\left(\overline{R}-1\right)$ out of the chain structure. Since the safe asset pays no interest and the investor discounts the future, he consumes the $\overline{R}-1$ immediately. An investor in state H_t, however, is necessarily on the end of an open chain. Thus, as long as he is not in an open one-link chain, he can reinvest \$1 in the project that did not fail for another period without expecting to incur a loss because, given his optimistic forecast, he expects the project to be fully funded.

An investor in state W_t after returns are realized at dates $t \geq 0$ has wealth of $2\overline{R}$ since he has not yet incurred a loss. The expected lifetime utility if he chooses at t to continue to hold a diversified portfolio is derived in Section IV-C. He could, instead, deviate to a safe portfolio. As was true in state H_t, he does this by pulling his funds out of the chain structure, either consuming them immediately or consuming some and reinvesting the rest in the safe asset.

Alternatively, an investor can deviate to a part-safe-part-risky portfolio, pulling \$1 out of the chain structure and reinvesting \$1 in one of the projects that he previously held. Since the safe asset pays no interest and the investor discounts the future, he consumes immediately what he does not reinvest $\left(2\overline{R}-1\right)$. Given his optimistic forecast, he anticipates that his coinvestors maintain their portfolios until they experience losses. This means that if he is on the interior of an open chain or in a closed chain, he expects his project to receive full funding and that he receives a return of $\overline{R}$ on his investment, which yields an expected lifetime utility of $V_{t+1}\left(f^i,\tilde{f}^{-i}|H_{t+1},k\right)$. If he is borderline in an open chain, then one of his former coinvestors has already incurred a loss and will not reinvest, while the other still has his whole initial endowment. Consequently, there is a 50 percent chance that he reinvests in the project in which his former coinvestor will also reinvest, again receiving an expected return of $\overline{R}$ and expected lifetime utility of $V_{t+1}\left(f^i,\tilde{f}^{-i}|H_{t+1},k\right)$. But there is also a 50 percent chance that he chooses the project in which he will be the only investor. In this case, he receives a return of zero on his \$1 investment and the expected lifetime utility $V_{t+1}\left(f^{i},\tilde{f}^{-i}|Z_{t+1},k\right)$. And of course, if he is in an open two-link chain and in state W_t, then he necessarily is a borderline investor with respect to both projects and thus necessarily gets $V_{t+1}\left(f^{i},\tilde{f}^{-i}|Z_{t+1},k\right)$.

Finally, an investor can deviate to an undiversified portfolio. To do so, he must consume his interest $\left(u\left(2\bar{R}-2\right)\right)$ at t and reinvest his remaining \$2 in one of the two assets that he held previously. With an optimistic forecast, he expects that his former coinvestor in the asset of choice will continue to invest \$1 until he experiences a loss. This implies that his expected return at $t + 1$ if he is on the interior of an open chain or in a closed chain is $\frac{4\bar{R}}{3}$, since a total of \$3 will be invested in the project, two of them the investor's. His coinvestor will receive $\frac{2\bar{R}}{3}$ on his own \$1 investment in the project, and either $\bar{R}$ or zero on his other investment, depending on whether it receives sufficient funding. The total return on the coinvestor's portfolio is thus either $\frac{2\bar{R}}{3}$ or $\frac{5\bar{R}}{3}$. The upper bound on $\bar{R}$ is assumed for simplicity to be 1.2 so that in the former case the coinvestor does not have \$1 to reinvest, and in the latter case can reinvest at most \$1. When the coinvestor does reinvest, it is a best response for him not to reinvest in the project formerly shared with the investor if he thinks the investor will not be reinvesting. And it is a best response for the investor to consume his $\frac{4\bar{R}}{3}$ immediately, rather than consume his interest and reinvest \$1, if he thinks his coinvestor will not reinvest. This is the equilibrium outcome studied, so the investor simply consumes his entire $\frac{4\bar{R}}{3}$ immediately. If, instead, he is borderline in an open chain, there is a 50 percent chance at date t that he chooses to reinvest in the asset for which he has a coinvestor, and a 50 percent chance that he ends up as the sole investor. He expects to receive the same lifetime return in the former case as if he were on the interior of the chain. In the latter case, where he is the only investor in the project of choice, the project receives exactly \$2 in funding, and he finds himself with wealth of $2\bar{R}$ at $t + 1$, which yields an expected lifetime utility of $\frac{u\left(2\bar{R}-2\right)}{1-b}$. This completes the derivation of $V_t\left(f^{i},\tilde{f}^{-i}|W_t,k\right)$, depicted in (4):

$$V_t\left(f^i,\tilde{f}^{-i}|W_t,k\right)=\begin{cases}\max\left\{u(2\overline{R}),u(2\overline{R}-1)+bu(1),u(2\overline{R}-2)+bu(2),u(2\overline{R}-2)+bu(1)+b^2u(1)\right\} \\ \quad \text{if } f_t^i(W_t,k)=S, \\ u(2\overline{R}-2)+b\left\{\sum_{r=3}^{k}\left[P\left(N_{r\,t+1}|W_t,k\right)\left(\frac{2}{r-1}\right)V_{t+1}\left(f^i,\tilde{f}^{-i}|H_{t+1},k\right)\right]\right. \\ \quad +\left[\sum_{r=3}^{k}P\left(N_{r\,t+1}|W_t,k\right)\left(1-\frac{2}{r-1}\right)+\sum_{r=2}^{k}P\left(C_{r\,t+1}|W_t,k\right)\right]V_{t+1}\left(f^i,\tilde{f}^{-i}|W_{t+1},k\right) \\ \quad \left.+P\left(N_{2\,t+1}|W_t,k\right)V_{t+1}\left(f^i,\tilde{f}^{-i}|Z_{t+1},k\right)\right\} \text{ if } f_t^i(H_t,k)=D, \\ u(2\overline{R}-1)+b\left\{\sum_{r=3}^{k}\left[P\left(N_{r\,t+1}|W_t,k\right)\left(1-\frac{2}{r-1}\right)+0.5P\left(N_{r\,t+1}|W_t,k\right)\left(\frac{2}{r-1}\right)\right.\right. \\ \quad \left.+\sum_{r=2}^{k}P\left(C_{r\,t+1}|W_t,k\right)\right]V_{t+1}\left(f^i,\tilde{f}^{-i}|H_{t+1},k\right) \\ \quad \left.+P\left(N_{2\,t+1}|W_t,k\right)V_{t+1}\left(f^i,\tilde{f}^{-i}|Z_{t+1},k\right)\right\} \text{ if } f_t^i(H_t,k)=P, \\ u(2\overline{R}-2)+b\left\{\left[\sum_{r=3}^{k}P\left(N_{r\,t+1}|W_t,k\right)\left(1-\frac{2}{r-1}\right)+\sum_{r=2}^{k}P\left(C_{r\,t+1}|W_t,k\right)\right]u\left(\frac{4\overline{R}}{3}\right)\right. \\ \quad \left.+\sum_{r=2}^{k}P\left(N_{r\,t+1}|W_t,k\right)\left(\frac{2}{r-1}\right)\left[\frac{1}{2}\frac{u\left(2\overline{R}-2\right)}{1-b}+\frac{1}{2}u\left(\frac{4\overline{R}}{3}\right)\right]\right\} \\ \quad \text{if } f_t^i(H_t,k)=U.\end{cases} \tag{4}$$

Appendix B

This appendix proves that there exist preferences and an open set of the parameter space for which Properties 1 and 2 are in fact consistent with the equilibrium considered in Section IV. The approach taken is to conjecture that there is an equilibrium characterized by the properties and to show that an investor would never choose to deviate from the behavior described in the properties.

Property 2 pertains to an investor who after returns are realized at any date $t \geq 0$ finds himself in state H_t with wealth of $\overline{R}$ because one of the projects he invested in failed. Appendix A described the actions available to an investor in this state. The property states that such an investor prefers not to reinvest his remaining principal of \$1 in a risky asset, which means that in equation (3) the expected utility from a safe portfolio dominates that from a part-safe-part-risky portfolio. The upper bound on the expected utility from a part-safe-part-risky portfolio is derived by assuming that the investor does not incur a loss due to contagion from reinvesting in the risky asset. It is thus

$u(\overline{R}-1)+bu(\overline{R})$ if the investor reinvests for one period and then consumes everything, and $u(\overline{R}-1)+b\left(u(\overline{R}-1)/(1-b)\right)$ if he instead consumes his interest and reinvests his dollar forever.[23]

For preferences $u(c)=(c-a)^{1-s}$ for $c > a$ and $u(c) = 0$ otherwise, and for parameter values s = 0.20, a = 0.03, $\overline{R}$ =1.05, b = 0.95, p = 0.01, the utility from consuming all wealth immediately, $u(\overline{R})$, is 1.016, while the utility from consuming the interest and investing $1 in the safe asset for one period is 0.971. The upper bound expected utility $u(\overline{R}-1)+bu(\overline{R})$ is 1.009, while the upper bound $u(\overline{R}-1)/(1-b)$ is 0.875. Consequently, the alternative of consuming all wealth immediately dominates reinvesting in a risky asset for the preferences and parameter values specified (i.e., $V_t\left(f^{\,i},\tilde{f}^{\,-i}|H_t,k\right)$ = $u(\overline{R})$ = 1.016), even in the best-case scenario, where there is no contagion risk associated with reinvesting in the risky asset, so Property 2 is satisfied. Since the property is satisfied with strict inequality, by continuity there exists an open set of parameter values for which it holds.

Property 1 addresses an investor's ranking of portfolios at date –1, before any shocks are realized. It states that the expected lifetime utility at date –1 from holding a maximum-return diversified loan portfolio at date 0 dominates that from deviating to any alternative portfolio. The expected utility at date –1 achievable from deviating to an undiversified portfolio at date 0 is calculated by considering what happens if the investor takes one of his dollars and invests it in the project in which his other dollar is already invested, while everyone else maintains their maximum-return diversified portfolios. With probability p, the project fails and he gets $u(0)$. With probability $1-p$, the project succeeds and he receives a return of $2\overline{R}/3$ per dollar at date 0 because his coinvestor will have continued to invest $1 in the project. His total return is thus $4\overline{R}/3$. As explained in Appendix A regarding the expected utility from deviating to an undiversified portfolio at any date after the shocks have been realized, he will consume his entire return immediately. The expected lifetime utility from deviating to an undiversified portfolio is thus

$$V_{-1}^{U}(k)=b\left\{pV_0\left(f^{\,i},\tilde{f}^{\,-i}|Z_0,k\right)+(1-p)u\left(\frac{4\overline{R}}{3}\right)\right\},$$

where $V_0\left(f^{\,i},\tilde{f}^{\,-i}|Z_0,k\right)$ is from equation (2). Thus, for the preferences and parameter values shown above to be consistent with Property 2, $V_{-1}^{U}(k)$ = 1.21.

[23] For any parameter values, if $u(\overline{R})<u(\overline{R}-1)+bu(\overline{R})$, then $u(\overline{R})<u(\overline{R}-1)/(1-b)$. Thus, if the investor prefers to roll over his investment for one period, then it is optimal for him to roll over his investment forever.

A second alternative is for the investor to deviate to a safe portfolio. That involves removing both dollars from the chain structure. The investor has the option of consuming both dollars immediately, or consuming one and investing the other in the safe asset for one period. The lifetime utility from this portfolio is

$$V_{-1}^{S} = b\left\{\max\left\{u(2),(1+b)u(1)\right\}\right\}.$$

For the preferences and parameter values used previously, V_{-1}^{S} = 1.808.

The final alternative is for the investor to deviate to a part-safe-part-risky portfolio at date 0. This deviation requires the investor to keep \$1 invested in one risky asset and to move \$1 to the safe asset at the start of date 0. With probability p, the project fails and the investor gets just the utility from his safe investment, $u(1)$. With probability $1-p$, the project succeeds and he receives an expected return of $\bar{R}$, given his optimistic forecast that other investors remain invested until they experience a loss, plus a return of \$1 on his safe investment. The investor has three options. He can immediately consume his wealth, $\bar{R}+1$. Alternatively, he can consume $\bar{R}$ and reinvest \$1 in the risky asset for another period, which yields a return of $a_1 V_1\left(f^i,\tilde{f}^{-i}|H_1,k\right)$, where

$$a_1 = \sum_{r=3}^{k}\left[P\left(N_{r1}|W_0,k\right)\left(1-\frac{2}{r-1}\right)+P\left(N_{r1}|W_0,k\right)\left(\frac{2}{r-1}\right)\frac{1}{2}\right]+\sum_{r=2}^{k}P\left(C_{r1}|W_0,k\right),$$

the probability that the investor's project pays a return of $\bar{R}$. As discussed in Appendix A, this probability is the probability that he is in a closed chain or on the interior of an open chain, in which case he necessarily has a coinvestor, plus the probability that he is borderline in an open chain and chooses (with probability ½) to reinvest in the project that gives him a positive return. His third option is to consume only his interest $\left(\bar{R}-1\right)$ and reinvest \$2 in risky assets at date 1. Because at dates 0 and later an investor does not know what projects are available other than those in his own portfolio, and because he only had one project in his portfolio at date 0, if he reinvests \$2 at date 1, then he must invest both dollars in the only project he held at date 0. That is, he must hold an undiversified portfolio. If his coinvestor reinvests also, which occurs if the coinvestor's other project was not hit by a shock, then his return is $4\bar{R}/3$, which he consumes immediately for the reasons discussed in Appendix A. If his coinvestor experienced a loss, either due to a shock (which occurred with probability p) or because he was in a closed two-link chain initially and the investor's deviation to a part-safe-part-risky portfolio left his project insufficiently funded, then the coinvestor will not reinvest by Property 2. This means that the investor's \$2 will be the only funds invested in the project, so the project provides a lifetime return of $\dfrac{u(2\overline{R}-2)}{1-b}$. Thus, his date –1 expected lifetime utility from deviating to a part-safe-part-risky portfolio is

$$V_{-1}^{P}(k) \le b\Big\{pu(1)+(1-p)\max\Big\{u(\overline{R}+1),u(\overline{R})+ba_{1}V_{1}\big(f^{i},f^{-i}|H_{1},k\big),$$
$$u(\overline{R}-1)+b\left(\big(p+P(C_{21}|W_{1},k)\big)\frac{u(2\overline{R}-2)}{1-b}+(1-p)u\left(\frac{4\overline{R}}{3}\right)\right)\Big\}\Big\}.$$

As shown above, for the preferences and parameter values being used, $V_{t}\big(f^{i},\tilde{f}^{-i}|H_{t},k\big)=u(\overline{R})$. Thus, $V_{-1}^{P}(k)$ may be rewritten as

$$V_{-1}^{P}(k) \le b\Big\{pu(1)+(1-p)\max\Big\{u(\overline{R}+1),u(\overline{R})+ba_{1}u(\overline{R}),$$
$$u(\overline{R}-1)+b\left(\big(p+P(C_{21}|W_{1},k)\big)\frac{u(2\overline{R}-2)}{1-b}+(1-p)u\left(\frac{4\overline{R}}{3}\right)\right)\Big\}\Big\}.$$

Finally, the expected lifetime utility at date –1 from a maximum-return diversified loan portfolio is

$$V_{-1}^{D}(k)=b\Big\{p^{2}V_{0}\big(f^{i},\tilde{f}^{-i}|Z_{0},k\big)+2p(1-p)V_{0}\big(f^{i},\tilde{f}^{-i}|H_{0},k\big)$$
$$+(1-p)^{2}\max\big\{V_{0}^{D}(k),V_{0}^{U}(k),V_{0}^{P}(k),V^{S}\big\}\Big\}$$
$$\ge b\Big\{2p(1-p)V_{0}\big(f^{i},\tilde{f}^{-i}|H_{0},k\big)+(1-p)^{2}\max\big\{V^{S},V_{0}^{P}(k)\big\}\Big\}.$$

Since $V_{t}\big(f^{i},\tilde{f}^{-i}|H_{t},k\big)= u(\overline{R})$ = 1.016 and V^{S} = $\max\big\{u(2\overline{R}),u(2\overline{R}-1)+bu(1),u(2\overline{R}-2)$ $+bu(2),u(2\overline{R}-2)+bu(1)+b^{2}u(1)\big\}$ = $u(2\overline{R}-1)+bu(1)$ = 1.983 for the preferences and parameter values specified above, $V_{-1}^{D}(k)\ge 1.865$. This lower bound clearly exceeds V_{-1}^{S} and $V_{-1}^{U}(k)$, so the investor does not have an incentive to deviate to a safe or undiversified portfolio.

It remains to show that the investor has no incentive to deviate to a part-safe-part-risky portfolio. There are three cases.

Case 1. $u(\overline{R}+1)=\max\left\{u(\overline{R}+1),u(\overline{R})+ba_{1}u(\overline{R}),u(\overline{R}-1)+b\left(\big(p+P(C_{21}|W_{1},k)\big)\dfrac{u(2\overline{R}-2)}{1-b}\right.\right.$ $\left.\left.+(1-p)u\left(\dfrac{4\overline{R}}{3}\right)\right)\right\}.$

In this case, given the preferences and parameter values used above, $V_{-1}^{P}(k)\le 1.66$, which is dominated by the lower bound on the expected utility from remaining diversified.

Case 2. $u(\bar{R}-1)+b\left(\left(p+P(C_{21}|W_1,k)\frac{u(2\bar{R}-2)}{1-b}+(1-p)u\left(\frac{4\bar{R}}{3}\right)\right)\right)=\max\{u(\bar{R}+1),u(\bar{R})$

$+ba_1u(\bar{R}),u(\bar{R}-1)+b\left(\left(p+P(C_{21}|W_1,k)\right)\frac{u(2\bar{R}-2)}{1-b}\right)+(1-p)u\left(\frac{4\bar{R}}{3}\right)\}.$

Again using the preferences and parameter values from above, $V_{-1}^{P}(k)$ < 1.865, and thus dominated by the lower bound on the expected utility from remaining diversified, if $P(C_{21}|W_1,k)<e$, where e < 0.308. Whether the restriction on e holds depends on the functional form of $q(2,k)$ and the value of k. For priors satisfying $\lim_{k\to\infty} q(ak,k)=1$ for a > 0, e is definitely be small enough for k sufficiently large.

Case 3. $u(\bar{R})+a_1u(\bar{R})=$ $\max\{u(\bar{R}+1),u(\bar{R})+ba_1u(\bar{R}),u(\bar{R}-1)+$

$b\left(\left(p+P(C_{21}|W_1,k)\right)\frac{u(2\bar{R}-2)}{1-b}\right)+(1-p)u\left(\frac{4\bar{R}}{3}\right)\}.$

The lower bound on $V_{-1}^{D}(k)$ can be written as

$$V_{-1}^{D}(k)\geq b\{2p(1-p)V_0(f^i,\tilde{f}^{-i}|H_0,k)+(1-p)^2V_0^P(k)\}$$
$$=b\{2p(1-p)u(\bar{R})+(1-p)^2\{u(2\bar{R}-1)+ba_1u(\bar{R})\}\},$$

making use of equation (3) and the fact that $V_t(f^i,\tilde{f}^{-i}|H_t,k)=$ $u(\bar{R})$ for the preferences and parameter values in question. To show that $V_{-1}^{D}(k)$ > $V_{-1}^{P}(k)$, then, it must be shown that

$$(1-p)^2u(2\bar{R}-1)>pu(1)+u(\bar{R})(1-p)\{1+ba_1-2p-(1-p)ba_1\}.$$

Observe that $1+ba_1-2p-(1-p)ba_1=1-2p+pba_1<1$ since $pba_1<2$. Hence, it suffices to show that

$$(1-p)^2u(2\bar{R}-1)>pu(1)+u(\bar{R})(1-p).$$

For the preferences and parameter values used above, the left side is 1.035, while the right side is 1.016. Therefore, the lower bound on the expected lifetime utility from remaining diversified exceeds the upper bound on the expected lifetime utility from deviating to a part-safe-part-risky portfolio.

It follows that Property 1 holds for the same preferences and parameters shown to be consistent with Property 2. Since the conditions required for Property 1 to hold are all satisfied with strict inequality, by continuity there exists an open set of parameter values for which the property holds. ■

Appendix C

This appendix proves Proposition 2. Let $a \in \{D, U, P, S\}$. Fix a date $t \geq 0$. Define $p^a(r)$ as the continuation utility of being in an open r-link chain after choosing action a, and $\bar{p}^a = \lim_{r\to\infty} p^a(r)$ for all $a = D, U, P, S$.

The idea of the proof is as follows. To show that $t(k)$ is decreasing in k for k sufficiently large, it suffices to show that $p^D(r) < p^D(r+2)$ and that for each date t and each action $a = U, P, S$,

$$V_t^D(k) - V_t^a(k) \tag{5}$$

is decreasing in k for $k \geq k'$. This can be established by showing that $V_t^D(k) - V_t^a(k)$ satisfies a stochastic dominance property. Specifically, observe that (5) can be rewritten in the form

$$K + b\left[\sum_{r\geq 2} P(N_{rt}|W,k)\left(p^D(r) - p^a(r)\right) + \sum_{r\geq 2} P(C_{rt}|W,k)\left(\bar{p}^D - \bar{p}^a\right)\right], \tag{6}$$

where K is a constant—for example, $K = u(2\bar{R}-2) - u(2\bar{R}-1)$ if $a = P$ for the preferences and parameter values used in Appendix B. Observe that

$$p^a(r) = \begin{cases} \dfrac{2}{r-1}u(\bar{R}) + \left(\dfrac{r-3}{r-1}\right)\left(u(2\bar{R}-2) + bp^a(r-2)\right) & \text{if } a = D \text{ and } r \geq 3, \\ \max\left\{\mathrm{u}(2\bar{R}), u(2\bar{R}-1) + bu(1), u(2\bar{R}-2) + bu(2), u(2\bar{R}-2) + bu(1) + b^2u(1)\right\} & \text{if } a = S \text{ and } r \geq 3, \\ \dfrac{r-2}{r-1}u(\bar{R}) & \text{if } a = P \text{ and } r \geq 3, \\ \dfrac{r-2}{r-1}u\left(\dfrac{4\bar{R}}{3}\right) + \dfrac{1}{r-1}\dfrac{u(2\bar{R}-2)}{1-b} & \text{if } a = U \text{ and } r \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, it suffices to show that

$$\sum_{r\geq 2} P(N_{rt}|W,k)\left(p^D(r) - p^a(r)\right) + \sum_{r\geq 2} P(C_{rt}|W,k)\left(\bar{p}^D - \bar{p}^a\right), \tag{7}$$

where $\bar{p}^D - \bar{p}^a = \lim_{r\to\infty}\left(p^D(r) - p^a(r)\right)$, is decreasing. This is shown below in two steps:

Step 1. For each $a = U, P, S$,

$$p^D(r) - p^a(r) \geq p^D(r-2) - p^a(r-2).$$

for r = 2,4,6,…,∞ or r = 1,3,5,…,∞.

Step 2. $P(\cdot|W,k)$ first-order stochastically dominates $P(\cdot|W,k+1)$ if $k \geq k'$. That is,

$$\forall j, \sum_{r=2}^{j} P(N_{rt}|W,k) \text{ is increasing in } k.$$

As a preliminary step, the following lemma is proven.

LEMMA. $p^D(r) \geq p^D(r-2)$.

PROOF. The proof proceeds by induction. Let $u^1 = u(\overline{R})$ and $u^2 = u(2\overline{R}-2)$. Then

$$p^D(r) = \frac{2}{r-1}u^1 + \left(\frac{r-3}{r-1}\right)(u^2 + bp^D(r-2)), \text{ for } r \geq 3.$$

Observe that $p^D(2) = 0$ and $p^D(4) = \frac{2}{3}u^1 + \left(\frac{1}{3}\right)u^2$, so $p^D(4) > p^D(2)$. Also, $p^D(3) = u^1$, $p^D(5) = \frac{1}{2}p^D(3) + \frac{1}{2}(u^2 + bp^D(3))$, and $p^D(5) - p^D(3) = \frac{1}{2}(u^2 - (1-b)p^D(3))$, which is greater than zero if $\frac{u^2}{u^1} > 1-b$. This last inequality holds for b sufficiently close to one.

Suppose that $p^D(r-2) > p^D(r-4)$ for $r \geq 5$, and show that $p^D(r) > p^D(r-2)$. Observe that $p^D(r-2) - p^D(r-4) > 0$ implies that $\left(\frac{2}{r-3}\right)u^1 + \left(\frac{r-5}{r-3}\right)u^2 + \left(b\left(\frac{r-5}{r-3}\right) - 1\right)p^D(r-4) > 0$, which implies that

$$2u^1 + (r-5)u^2 + (b(r-5)-1)p(r-4) > 0. \tag{8}$$

Now consider whether $p^D(r) > p^D(r-2)$. This inequality holds iff

$$2u^1 + (r-3)u^2 + (b(r-3)-1)p^D(r-2) > 0. \tag{9}$$

Hence, it remains to verify that (9) holds. Observe that

$$\begin{aligned} &2u^1 + (r-3)u^2 + (b(r-3)-1)p^D(r-2) \\ &> 2u^1 + (r-3)u^2 + (b(r-3)-1)p^D(r-4) \\ &> 2u^1 + (r-5)u^2 + (b(r-5)-1)p^D(r-4) \\ &> 0 \end{aligned}$$

by (8). Hence, (9) holds.■

PROOF OF STEP 1. Again let $u^1 = u(\overline{R})$ and $u^2 = u(2\overline{R}-2)$. There are three cases:

Case 1: $p^D(r) - p^U(r) \geq p^D(r-2) - p^U(r-2)$.

Proof. Let $v^U \equiv u\left(\frac{4\overline{R}}{3}\right)$. Observe that for b sufficiently close to one, $\frac{u(2\overline{R}-2)}{1-b} > v^U$. For such values of b, $p^U(r) = \frac{r-2}{r-1}v^U + \frac{1}{r-1}\frac{u(2\overline{R}-2)}{1-b}$, which is decreasing in r. It follows that $p^U(r-2) > p^U(r)$. Using this result along with the result in the Lemma (i.e., $p^D(r) > p^D(r-2)$) establishes Case 1.

Case 2: $p^D(r) - p^P(r) \geq p^D(r-2) - p^P(r-2)$.

Proof. Observe that $p^P(r) = \left(\frac{r-2}{r-1}\right)u(\overline{R})$. Thus, $p^D(r) - p^P(r) = M(r) \equiv$ $\frac{4-r}{r-1}u^1 + \left(\frac{r-3}{r-1}\right)\left(u^2 + bp^D(r-2)\right)$. Multiplying through by $r-1$, for r odd, yields $(r-1)M(r) = (4-r)u^1 + (r-3)\left(u^2 + bp^D(r-2)\right)$. Hence, to show $M(r) - M(r-2)$, or $(r-1)M(r) - (r-1)M(r-2)$, it suffices to show that

$$\begin{aligned}
&(4-r)u^1 + (r-3)\left(u^2 + bp^D(r-2)\right) \\
&> (6-r)u^1 + (r-5)\left(u^2 + bp^D(r-2)\right) \\
&> (6-r)u^1 + (r-5)\left(u^2 + bp^D(r-4)\right).
\end{aligned}$$

Clearly, the last inequality must hold since $p^D(r-2) > p^D(r-4)$ by the Lemma. It remains to verify that the first inequality holds. This can be rewritten as $2\left[u^2 - u^1 + bp^D(r-2)\right] > 0$. Since $r \geq 3$, if r odd, it is enough to show (since $p^D(r) > p^D(r-2)$ by the Lemma, $2\left[u^2 - u^1 + bp(3)\right] > 0$. Since $p^D(3) = u^1$, this inequality can be rewritten as $2\left[u^2 + (b-1)u^1\right] > 0$, or $\frac{u^2}{u^1} > 1-b$.

Suppose, instead, that r even. Then it is enough to show that $2\left[u^2 - u^1 + bp(4)\right] > 0$. Since $p(4) = \frac{2}{3}u^1 + \left(\frac{1}{3}\right)u^2$, this can be rewritten as $2\left[u^2 - u^1 + b\left(\frac{2}{3}u^1 + \frac{1}{3}u^2\right)\right] > 0$, or

$2\left(\frac{2b}{3}-1\right)u^1+2\left(1+\frac{b}{3}\right)u^2>0$. This last inequality implies $\frac{u^2}{u^1}>\frac{1-(2b/3)}{1+(b/3)}$.

Therefore, when the inequalities $\frac{u^2}{u^1}>1-b$ and $\frac{u^2}{u^1}>\frac{1-(2b/3)}{1+(b/3)}$ are satisfied, whether *r* even or odd, $M(r)$ increases.

Case 3: $p^D(r)-p^S(r)\geq p^D(r-2)-p^S(r-2)$.

Proof. Deviating to a safe portfolio yields utility of $\max\{u(2\overline{R}),u(2\overline{R}-1)+bu(1),$ $u(2\overline{R}-2)+bu(2),u(2\overline{R}-2)+bu(1)+b^2u(1)\}$, which is a constant, independent of *r*. Thus, the condition $p^D(r)-p^S(r)\geq p^D(r-2)-p^S(r-2)$ reduces to $p^D(r)\geq p^D(r-2)$, which is true by the Lemma.■

PROOF OF STEP 2. To use a first-order-stochastic-dominance property, it must be shown that any left tails of the distributions of both even-numbered expected utilities $\left(p^D(2r)-p^a(2r)\right)$ and odd-numbered expected utilities $\left(p^D(2r+1)-p^a(2r+1)\right)$ in (6) are separately increasing in *k*. Specifically, it must be shown that there is some k' such that if $k\geq k'$, then

$$\forall\ j\leq k,\quad \sum_{r=2}^{j}P\left(N_{2r\,t}|W_t,k\right)\quad \text{is increasing in } k \tag{10}$$

and

$$\forall\ j\leq k,\quad \sum_{r=2}^{j}P\left(N_{2r+1\,t}|W_t,k\right)\quad \text{is increasing in } k\,. \tag{11}$$

The proof that (10) holds is as follows; the argument for (11) is completely analogous. Fix some *j* with $j\leq k$. Observe that

$$\begin{aligned}\sum_{r=2}^{j}P\left(N_{rt}|E_t(k)\right)&=\frac{\sum_{r=t}^{j+2(t-1)}\overline{P}\left(N_{2r1}|(W_1,k)\right)}{\Lambda_t(k)}\\&=\frac{\sum_{r=t}^{j+2(t-1)}\overline{P}\left(N_{2r1}|(W_1,k)\right)}{\sum_{r=2t}^{k}\overline{P}\left(N_{r1}|(W_1,k)\right)+\sum_{r=2}^{k}\overline{P}\left(C_{rt}|W_t,k\right)}\\&=\frac{\sum_{r=t}^{j+2(t-1)}(2r-1)p(1-p)^{2r-2}\left[q(2r+1,k)+p\left(\sum_{m=2r+2}^{k}q(m,k)\right)\right]}{\sum_{r=2t}^{k}(r-1)p(1-p)^{r-2}\left[q(r+1,k)+p\left(\sum_{m=r+2}^{k}q(m,k)\right)\right]+\sum_{r=2}^{k}(1-p)^{r-2}q(r,k)}.\end{aligned}$$

Let

$$G(k) = \sum_{r=t}^{j+2(t-1)} (2r-1)p(1-p)^{2r-2} q(2r+1,k),$$

$$H(k) = \sum_{r=t}^{j+2(t-1)} (2r-1)p^2(1-p)^{2r-2} \sum_{m=2r+2}^{k} q(m,k),$$

$$I(k) = \sum_{r=2t}^{k} (r-1)p(1-p)^{r-2} q(r+1,k),$$

$$J(k) = \sum_{r=2t}^{k} (r-1)p^2(1-p)^{r-2} \sum_{m=r+2}^{k} q(m,k),$$

and

$$L(k) = \sum_{r=2}^{k} (1-p)^{r-2} q(r,k).$$

Observe that, given the assumptions on the distribution $q(r,k)$, $G(k)$ m 0, $H(k)$ **k** H, $I(k)$ m 0, $J(k)$ **k** J, and $L(k)$ m 0 as $k \to \infty$, and that H and J are positive constants such that $H/J < 1$. Moreover, $L(k)/G(k) \to 0$ as $k \to \infty$. That is, $L(k) = o(G(k))$.[24] Finally, $I(k) + J(k)$ is decreasing in k. To see this, recall that $I(k) + J(k) \to J$. Also, $J(k) \geq J - p^2 k(1-p)^{k+1}$, while $I(k) \geq p(ak-1)(1-p)^{ak} q(ak,k)$. But $J - p^2 k(1-p)^{k+1} + p(ak-1)(1-p)^{ak} q(ak,k) > J$ for large enough k, and the left side of this inequality converges to J as $k \to \infty$. Hence, $I(k) + J(k) > J$ and so $I(k) + J(k)$ m J. It follows that

$$\frac{G(k) + H(k)}{o(G(k)) + I(k) + J(k)} \ \mathbf{k} \ \frac{H}{J}.$$

That is,

$$\sum_{r=2}^{j} P(N_{rt} | W, k) \ \mathbf{k} \ \frac{H}{J} \text{ as } k \to \infty.$$

One can conclude, then, that for k sufficiently large, $P(\cdot | W_t, k)$ first-order stochastically dominates $P(\cdot | W_t, k+1)$ if $k \geq k'$.■

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[24] For any function f defined on some variable x, $f = o(x)$ if $\lim_{x\to\infty} \frac{f(x)}{x} = 0$. That is, f increases at rate x.

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