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Trend and uncertainty in the long-term real interest rate: Bayesian exponential tilting with survey data

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Abstract

Economic growth and inflation as well as interest rates have remained low in the U.S. since the Great Recession of 2007-9, creating renewed interest in the secular stagnation hypothesis that tightly links the permanent downward shift in the real interest rate with the predictable decline in economic growth and inflation. To empirically investigate this hypothesis, I set-up and estimate a time-varying parameter vectorautoregression (TVP-VAR) model with stochastic volatility using Bayesian methods to obtain trend estimates for the long-term real interest rate. I estimate trend as the long-run average forecast and simulate its predictive density. The median estimate of the trend long-term real interest rate is mostly in line with the existing literature. However, uncertainty band is generally higher. In addition, uncertainty is not in general symmetric with downside (upside) risk more prominent during certain periods. When model-implied forecasts are tilted to match variances as well as means of forecasts from the survey data, the trend estimates of the long-term nominal interest rate and inflation feature much smaller downside risk than those without tilting or tilting to match means only. Forecasts tilted to match both means and variances are more in line with the finding from the existing literature that the U.S. has experienced the negligible chance of a shift to deflation regime during the recent period. While point estimates of trends from the TVP-VAR model are roughly in line with the secular stagnation hypothesis, the level and time variation of uncertainty in the predictive density cast doubt on the idea that these declines were fairly predictable and permanent.

Key Words: Long-run real interest rate; Time-varying parameter vectorautoregression; Bayesian predictive density.
JEL Classification: C11, E43

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1 Introduction

The long-term interest rate in the U.S. and the world has substantially declined over the past three decades and has remained at a historically low level. For example, the 10-year U.S. Treasury yield decreased from 11.63 percent on January 2, 1985 to 2.34 percent on January 4, 2016. While inflation and inflation expectations also declined significantly during the same period, they do not entirely account for the decline by most measures. Thus, the long-term real interest rate might have declined too (CEA 2015). Since the long-term real interest rate is a key variable affecting various intertemporal choices such as borrowing for home purchases and the valuing investment projects, the downward shift in the trend long-term real interest rate attracted much attention. Lowered growth expectations due to slowing productivity growth and shifts in desired savings that are related to demographic changes are often cited as key factors driving the secular trend in the long-term real interest rate (Gagnon et al. (2016), Rachel and Smith (2015), and Summers (2016) for instance).[1] If this hypothesis were right, we would likely see fairly predictable declines in gorwht and inflation as well as the real interest rate.

In this paper, I examine if this secular stagnation hypothesis is supported by a flexible time-series model well suited to identify time-varying trends. I set up a time-varying parameter vectorautoregression (TVP-VAR) model to estimate the trend component in the long-term real interest rate. The TVP-VAR model includes stochastic volatility in each variable to capture movements in the trend and volatility of major macroeconomic and financial variables. Following Cogley and Sargent (2005) and Cogley, Morozov, and Sargent (2005), I estimate the TVP-VAR model using Bayesian methods and simulate the entire predictive density for trend to identify not only the point estimate but also higher-order moments such as volatility and skewness regarding the trend estimates. In addition, I measure time-varying trends from model estimation in multiple ways. The most common way to define the trend component in the TVP-VAR model is to calculate the model-implied long-run mean conditional on no future shocks to time-varying coefficients as done by Colgey and Sargent (2005), for example. By using Bayesian methods to incorporate parameter uncertainty, the method provides uncertainty bands of trend estimates based on the posterior distribution of parameters and latent states

[1] Del Negro et al. (2017) provide an alternative explanation that attributes a bulk of the decline into safety and liquidity premium commanded by the long-term US treasury bonds. They measure these premiums based on the spread between Treasury bond yields and corporate bond yields. However, Williams (2017) and Wu (2017) question if there are significant trends in these spreads to account for the decline in the real interest rate.

(time-varying coefficients in the VAR). However, this measure abstracts from future shock uncertainty, thereby underestimating the actual uncertainty surrounding trend estimates. As an alternative measure, I use long-horizon (longer than 10-year) forecasts from the model to take into account both parameter uncertainty and future shock uncertainty as in Amir-Ahmadi et al. (2016). To further isolate low-frequency movements, I use the 10-year average forecasts from 11 years to 20 years from the current period as used by Müller and Watson (2016). Furthermore, I tilt these long-horizon forecasts to match means and variance of the long-term forecasts for inflation and the nominal 10 year Treasury yield from the survey of profession forecasters (SPF). These forecasts consist of projected values for target variables averaged over 10 years from the current period. Krüger et al. (2017) show that tilting forecasts from a statistical model such as the VAR to match external forecasts from the survey data improves the accuracy of out-of-sample forecasts. In the context of this paper, survey data information might be useful for testing one of main implications of the secular stagnation hypothesis: that the decline in the real interest rate was largely predictable early on.

Based on the annual U.S. data from 1915 to 2016, the median estimate of the trend from the TVP-VAR model provides evidendce that the trend long-term real interest rate has declined since the 80s in line with the finding in the previous literature (Gagnon et al. (2016), Johannsen and Mertens (2016), CEA (2015), etc.).[2] However, I also find that the uncertainty band derived from the predictive density is wide enough to cast doubt on the view that the decline in the real interest rate was permanent and mostly predictable. This is true for all three measures of the trend real interest rate (the model-implied long-run mean, long-horizon average forecasts with and without tilting) estimated from the TVP-VAR model. Especially, when long-horizon forecasts are used to construct trend estimates, the magnitude of the uncertainty band signficantly increases.

To see the time-variation of tail risks, I look at higher order moments of trend estimates from long-horizon forecasts. Due to the nonlinearity related to time-varying coefficients, the TVP-VAR model generates non-zero skewness of long-run forecasts although all the shocks are drawn from symmetric normal distributions. Without tilting, the model-implied forecasts suggest a substantial downside risk for the long-run nominal interest rate and inflation since the Great recession of 2007-9 while risks are roughly balanced for these variables with tilting. Both forecasts suggest downside risk for the

[2] Gagnon et al. (2016) and Johannsen and Mertens (2016) focus on the delcine in the trend short-term real interest rate. As long as the term premium is stationary as typically assumed in the literature, the short-term real interest rate and the long-term interest rate share the same trend, meaning that the time-variation of trend can be interpreted in the same way.

trend long-term real interest rate near the end of sample but for different reasons. For estimates with tilting, the downside risk of the trend long-term real interest rate is driven by the upside risk of trend inflation. In contrast, for estimates without tilting, it is driven by the prominent downside risk of the long-term nominal interest rate. However, trend inflation estimates without tilting suggest implausibly high deflation risk in the U.S. since 1992, which is at odds with the finding that the risk of deflation has been modest in the U.S. except for a short period in 2009 (Arouba et al. (2017)). Interestingly, trend estimates with tilting to match only means of forecasts from the survey also imply implausibly high chances of deflation in the U.S. during the recent years, suggesting that the cross-sectional dispersion of forecasts provides additional external information to discipline the predictive density from the TVP-VAR model. When I utilize both means and variances from survey data to tilt long-horizon forecasts from the TVP-VAR model, trend estimates cast doubt on one of the main arguments of the secular stagantion hypothesis: specifically, that trend inflation and the real interest rate might be stuck at the low value with narrow uncertainty bands.

This paper proceeds as follows. Section 2 introduces the TVP-VAR model estimated in this paper and describes the data used. Section 3 examines predictive density for trend estimates and compares higher order moments of different trend estimates. Section 4 concludes.

2 A Time-varying Parameter Vectorautoregression Model of Long-term Interest Rates

The TVP-VAR model used in this paper includes five observed variables: real per-capita consumption growth (Δc_t), real per-capita dividend growth (Δd_t), CPI inflation (π_t), the 10 yearnominal interest rate ($y_{10,t}$), and the aggregate price-dividend ratio ($z_{d,t}$). Not only are VAR coefficients in the model time varying, but each variable in the model is also subject to a shock with time-varying volatility. I include Δd_t and $z_{d,t}$ because stock market data provide information about the expected future real discount rate that comoves with the long-term real interest rate.[3]

[3]Under Campbell-Shiller (1988) log-linear approximation, $\ln z_{d,t} \approx \frac{\kappa}{1-\rho} + \sum_{j=0}^{\infty} \rho^j E_t(\Delta d_{t+1+j} - r_{t+1+j})$ where r_{t+1+j} is the real discount rate at $t+1+j$, which is the sum of the real interest rate and the risk premium, where ρ and κ are constants determined by the steady-state log price-dividend ratio.

2.1 Data

The estimation uses long-run annual observations for the U.S. from 1891 and 2016. Observations from 1891 to 1914 are used only to construct prior distributions. The data can be downloaded from Robert Shiller's website (www.econ.yale.edu/ shiller/data.htm).[4] Given the low frequency of data, I estimate a first-order VAR model. Forecasts for CPI inflation and the 10 yearTreasury yield from the Survey of Professional Forecasters (SPF) are used to tilt model-implied forecasts toward observed proxies for expectations. Survey data for average expectations for inflation over ten years are available from 1992 while the expectaions of the 10 yearTreasury yield averaged over ten years are available from 1994.

2.2 A TVP-VAR model with stochastic volatility

I denote the vector of five variables by $Y_t = [\Delta c_t, \Delta d_t, \pi_t, y_{10,t}, \ln z_{d,t}]'$. The dynamics of Y_t in the TVP-VAR model can be described as follows:

$$\begin{aligned}
Y_t &= \theta_{0,t} + \theta_{1,t} Y_{t-1} + u_t\,,\; u_t = B\varepsilon_t \sim \mathcal{N}(0, \Sigma_t), \\
\theta_{0,i,t} &= \theta_{0,i,t-1} + \epsilon_{\theta_{0,i},t}\,,\; (i = 1, \cdots, 5), \\
\theta_{1,i,j,t} &= \theta_{1,i,j,t-1} + \epsilon_{\theta_{1,i,j},t}\,,\; (i, j = 1, \cdots, 5), \\
\Sigma_t &= (B^{-1})' D_t (B^{-1})\,,\; \ln D_{ii,t} = \ln D_{ii,t-1} + \sigma_{\xi,i}\xi_{i,t}\,,\; \xi_{i,t} \sim \mathcal{N}(0,1)\,,\; (i = 1, \cdots, 5), \\
\theta_t &= \text{vec}([\theta_{0,t}, \theta_{1,t}])\,,\; \epsilon_{\theta,t} = \text{vec}([\epsilon_{\theta_0,t}, \epsilon_{\theta_1,t}])\,,\; \Sigma_\theta = E(\epsilon_{\theta,t}\epsilon'_{\theta,t})\,,\; \epsilon_{\theta,t} \sim \mathcal{N}(0, \Sigma_\theta).
\end{aligned} \tag{1}$$

where D_t is a diagonal matrix whose off-diagonal terms are zero. The model estimates consist of the two parts. The first part is the collection of constant parameters ($\vartheta = [B, \Sigma_\theta, \sigma_{\xi,i}]$) and the second part is the collection of latent variables including time-varying coefficeints and time-varying volatilities ($S_t = [\theta_t, D_t]$).

[4]For consumption growth from 1930 to 2016, I use year-over-year growth rate of per capital real consumption expenditures available from Haver analytics while the earlier data draw on Kendrick's measures based on real consumption of non-durable goods and services. The 2016 observation for consumption growth is the growth rate of consumption from year 2014 to year 2015. This is done match the information set contained in interest rates and stock prices that are based on January values of the corresponding year. For dividend growth, I use the December-to-December growth rate in the previous year. For instance, 1992 dividend growth is calculated as the the growth from December 1990 to December 1991. Inflation is defined by the change in the price level from the January of the previous year to the January of the current year while price-dividend ratio and the 10 year Treasury yield are the January level of the corresponding year.

The Bayesian estimation approach adopted in this paper first constructs the prior distribution for ϑ and the intial estimates of latent variables S_0, $p(\vartheta, S_0)$. I use the pre-sample data from 1891 to 1914 to estimate a constant parameter VAR with constant volatility. The prior distributions for S_0 are obtained using coefficient estimates and VAR residual estimates from this pre-sample exercise.[5] Under the Bayesian approach, prior information is combined with the sample data to obtain the posterior distribution. In the constant parameter VAR model with constant volatility, we can compute the likelihood $p(Y^T|\vartheta)$ analytically and combine this information with conjugate priors to derive the posterior distribution of $p(\vartheta|Y^T)$ under the normality assumption of shocks.[6] Due to the fact that time-varying coefficients in the VAR make variables in the VAR nonlinear functions of the past shocks, it is not feasible to easily compute the likeihood function that integrates out unobserved latent state variables in the TVP-VAR model. In addition, time-varying volatilities generate additional nonlinearities. Following Cogley and Sargent (2005), we rely on the posterior simulation of ϑ and S^T using the property that the distribution of each subset in $(\vartheta, \theta_t, D_t)$ conditional on the rest is tractable for simulation. To exclude parameters implying the explosive dynamics, we throw away ϑ and S^T implying that Y_t does not follow a stationary process at any point of time. I generate 100,000 posterior draws of (ϑ, S^T) and burn-in the first 50,000 draws, using the remaining 50,000 draws for analysis.[7]

3 Predictive Density for Trend

This section describles how to obstain the predictive density for the long-term real interest rate from posterior draws of (ϑ, S^T). First, we have to measure the trend component in a way that can be derived from model estimates. Second, depending on the measure of the trend component, the simulation of future shocks might be necessary to obtain draws from the predicitve density.

[5] The details of prior elicitation and posterior simulation are explained in the appendix.

[6] Y^T stands for $[Y_1, \cdots, Y_T]$.

[7] For actual inference, I only save every 10th draw. Therefore, 5,000 draws are used for posterior inference discussed below.

3.1 Sources of uncertainty behind predictive densities for trend estimates

As illustrated by Cogley, Morozov, and Sargent (2005), the TVP-VAR model with time-varying volatility exhibits multiple uncertainties for long-horizon forecasts. Denote forecasts for $[Y_{t+1}, \cdots, Y_{t+h}]$ by Y_{t+1}^{t+h}. There are three different types of uncertainty behind the predictive density $p(Y_{t+1}^{t+h}|Y^t)$.

$$\begin{aligned} p(Y_{t+1}^{t+h}|Y^t) &= \underbrace{p(\vartheta, S^t|Y^t)}\ \underbrace{p(S_{t+1}^{t+h}|\vartheta, S^t)}\ \ \underbrace{p(Y_{t+1}^{t+h}|\vartheta, Y^t, S_{t+1}^{t+h})}, \\ &= (\text{parameter})(\text{VAR coefficients and volatility})(\text{VAR residual shocks}). \end{aligned} \tag{2}$$

One popular way of defining the trend in the literature is to take the model-implied long-run mean of Y_t conditional on no future shocks to time-varying coefficients.[8]

$$\bar{Y}_t = \lim_{h\to\infty} E_t(Y_{t+h}|\epsilon_{\theta,t+j} = 0\,,\ j = 1, \ldots, h) = \lim_{h\to\infty} \sum_{j=0}^{h} \theta_{1,t}^{j}\theta_{0,t} = (I - \theta_{1,t})^{-1}\theta_{0,t}. \tag{3}$$

Since this measure of trend depends only on the current estimates of $\theta_{0,t}$ and $\theta_{1,t}$, it does not require any simulation for future shocks and draws from the predictive density can be obtained rather quickly. However, it takes into account only parameter uncertainty and may underestimate uncertainty about trend due to future shocks to time-varying coefficients.

An alternative defition of trend as used by Amir-Ahmadi, Matthes, and Wang (2016) is a long horizon (h-step ahead) forecast. To better isolate low frequency movements in long horizon forecasts, I adopt the following the average of long horizon forecats as a meausre of the trend:[9]

$$\bar{Y}_{t,t+h+1:t+2h} = \frac{\sum_{j=1}^{h}(Y_{t+h+j})}{h}. \tag{4}$$

The predictive density of $Y_{t,t+h+1:t+2h}$ ($p(Y_{t,t+h+1:t+2h}|Y^t)$) can incorporate future shock uncertainty as well as parameter uncertainty. For h large enough like 10 years, this measure can be a good proxy for a long-term trend. I compute the ex-ante real interest rate ($r_{10,t}$) by subtracting the next 10 yearaverage inflation forecast from the forecast for

[8] To compute the long-run mean, I throw away draws of VAR coefficients that imply non-stationarity. In other words, I only consider θ_t that belongs to the stationary region Θ_s.

[9] Müller and Watson (2016) measure uncertainty about long-run predictions using the similar long-horizon avearage of forecasts. I again impose the stationarity restriction on parameter draws and consider long-term forecasts conditional on $\theta \subset \Theta_s$.

the 10 yearnominal interest rate. One can calculate the predicitve density by simulation from each posterior draw of ϑ and S^t.

Another advantage of using a finite but long-horizon forecast as a proxy for trend is that one can match implications of the model estimates with observed expectations from survey data. For example, the SPF provides long-run forecasts for CPI inflation and the 10 yearTreasury yield. Using the method explained by Robertson, Tallman, and Whitemann (2005), I can exponentially tilt the original predictive density for $\bar{Y}_{t,t+h+1:t+2h}$ to minimize the distance between the predictive moments computed from model estimates of $[\vartheta, S^T]$ and empirical moments from survey data observations. More specifically, let's assume that the orignial predicitve density is represented by N equally likely long-term forecasts $\bar{Y}^j_{t,t+h+1:t+2h}$ for $j = 1, \cdots, N$. Suppose survey data evidence for $\frac{\sum_{j=1}^h E_{t,i}(Y_{t+j})}{h} = Y^S_{t,t+h,i}$ is available for each forecaster i. One can find out the new predictive density $p^\star(\bar{Y}_{t,t+h+1:t+2h})$ by minimizing the distance between the two predictive densities based on the Kullback-Leibler information criterion (KLIC) subject to the restriction that means and variances of forecasts from the SPF are matched under the new predictive density. The problem can be formalized as follows:

$$
\begin{aligned}
p^\star(\bar{Y}_{t,t+h+1:t+2h}) = \text{argmin} K(p, p^\star) = \int \ln[\frac{p^\star(\bar{Y}_{t,t+h+1:t+2h}|Y^t)}{p(\bar{Y}_{t,t+h+1:t+2h}|Y^t)}]p^\star(\bar{Y}_{t,t+h+1:t+2h}|Y^t)dY^{t+2h}_{t+h+1}, \\
\text{subject to} \int \frac{\sum_{j=1}^h Y_{t+j}}{h} p^\star(\frac{\sum_{j=1}^h Y_{t+j}}{h}|Y^t)dY_{t+j} = E_t(Y^S_{t,t+h}), \\
\int (\frac{\sum_{j=1}^h Y_{t+j}}{h} - E_t Y^S_{t,t+h})^2 p^\star(\frac{\sum_{j=1}^h Y_{t+j}}{h}|Y^t)dY_{t+j} = V_t(Y^S_{t,t+h}),
\end{aligned} \tag{5}
$$

where $E_t(Y^S_{t,t+h})$ and $V_t(Y^S_{t,t+h})$ are cross-sectional means and variances of survey participants' forecasts. The solution to this problem is characterized by probability weights $p^\star_i$ for $(i = 1, \cdots, N)$, where $p^\star_i$ is defined by

$$
\begin{aligned}
p^\star_i &= \frac{e^{\gamma' Y^i_{t+h}}}{\sum_{i=1}^N e^{\gamma' Y^i_{t+h}}}, \\
\gamma &= argmin \sum_{i=1}^N e^{\gamma_1'(\frac{\sum_{j=1}^h Y^i_{t+j}}{h} - E_t(Y^S_{t,t+h})) + \gamma_2'((\frac{\sum_{j=1}^h Y^i_{t+j}}{h} - E_t(Y^S_{t,t+h}))^2 - V_t(Y_{t,t+h}))}, \\
\gamma &= [\gamma_1, \gamma_2].
\end{aligned} \tag{6}
$$

The new predictive density can be represented by resampling the existing forecasts using

$(p_1^\star, \cdots, p_N^\star)$ as new probability weights. The nonlinear dynamics of Y_t embedded in the TVP-VAR model generate the time-variation not only in point forecasts but also higher-order moments such as uncertainty and skewness in trend estimates. Under the random-walk hypothesis, the h-step ahead forecast error for Y_t contains the h-th order polynomial for $\epsilon_{\theta,t+1}$ because it involes $\prod_{j=1}^{h} \theta_{1,t+j}$. Hence, for $h \geq 2$, higher order moments of $Y_{t+h} - E_t(Y_{t+h})$ must include higher order moments of ϵ_{t+1}^h. This property implies that although ϵ_{t+1} itself is normally distributed, $Y_{t+h} - E_t(Y_{t+h})$ is not normally distributed for $h \geq 2$, generating non-zero skewness. For example, for $h = 2$, $Y_{t+2} - E_t(Y_{t+2})$ is characterized by the mixture of normal distributions and chi-square distributions. Having a non-normal distribution as a component allows the predictive density to generate time-varying non-zero skewness.[10]

3.2 Trend and uncertainty in the estimated long-term real interest rate

Recently, Krüger et al. (2017) show that tilting forecasts from the Bayesian VAR (BVAR) to match nowcasts from surveys improves the accuracy of both point and density forecasts. In particular, their finding suggests that tilting the BVAR forecasts based on both means and variances of nowcasts outside the BVAR yields slightly greater gains in accuracy than tilting just based on the means of nowcasts.[11] In a similar vein, I use means and variances of forecasts for CPI inflation and the nominal 10 year Treasury yield over the following ten-years from the SPF to tilt trend estimates from the TVP-VAR model. Figure 1 shows means and variances of CPI inflation and the nominal 10 year Treasury yield forecasts from the SPF from 1992 to 2016. While means of forecasts generally trended down, variances of forecasts exhibited occasional spikes without clear trends.

I generate draws from the following three predictive densities: 1) $p(\bar{Y}_t)$, 2) $p(\bar{Y}_{t,t+h+1:t+2h})$, and 3) $p^\star(\bar{Y}_{t,t+h+1:t+2h})$. Each predictive density is represented by posterior predictive draws. Figure 2 describes the median estimate of the model-implied unconditional mean of the long-term real interest rate with the corresponding interquartile range. This trend estimate moves very gradually declining from 2.7 percent in 1915 to 1.2 percent in 2016. However, the magnitude of the decline since 1980 is less notable, amounting to only 0.4 percent. This is much lower than the 1.3 percent decline documented in the literature

[10] The appendix illustrates this property by looking at a simple AR(1) example.

[11] While they calculate the variance from time-series of survey forecast errors, I use the cross-sectional disperson of forecasts to proxy the variance. Two measures are different but related in general as shown by Krüger and Nolte (2016).

for the same time period.[12]

Other trend estimates using long-horizon forecasts from the model generate the magnitude of the decline in the long-term real interest rate comparable to what is reported in the literature. Figure 3 shows the median trend estimate from the model-implied forecast with the interquartile range. The plot on the left represents the model-implied trend estimate. The trend estimate declines from slightly above 2 percent in 1992 to 1.2 percent in 2016, consistent with findings in other papers (Gagnon et al. (2016), Johansen and Mertens (2016), and Del Negro et al. (2017), etc.). However, the uncertainty band is much wider than reported in the literature, with the interquartile range of the trend estimate of the long-run real interest rate between -2 percent and slightly above 4 percent as of 2016. For comparison, the length of the 95% band for the natural real interest rate in Del Negro et al. (2017) is about 1.6 percent, substantially smaller than that of the uncertainty band obtained in the TVP-VAR model.[13]

As discussed in the previous part, $\bar{Y}_{t,t+h+1:t+2h}$ tends to have a much wider uncertainty band because it accounts for uncertainty due to future shocks as well as uncertainty regarding current estimates of ϑ and S^t. Indeed, the current estimate of the interquartile range of the trend real interest rate is wide enough to contain almost all the median trend real interest rate estimates during the entire sample period between 1915 and 2016. This is not surprising that the long-term interest rate is highly persistent, amplyfying the forecast uncertainty as the horizon lengthens. Nonetheless, the median estimate of $\bar{Y}_{t,t+h+1:t+2h}$ is mostly aligned with point estimates from the existing literature that find the current level of the trend real interest rate between 1 and 1.5 percent (quasi real-time estimates Johansen and Mertens (2016), for example).[14]

To compare the impacts of exponential tilting on trend estimates, Figure 4 plots median trend estimates from the TVP-VAR model with and without tilting. I include the ex-post measure of the real interest rate reported in CEA (2015) that compute the real 10 year interest rate by subtracting the five-year moving average of current and

[12]For example, see the magnitude of the decline in the natural real interest rate in Del Negro et al. (2017).

[13]Even the long-run mean measure of trend from the TVP-VAR model shows a wider uncertainty band than reported in Del Negro et al. (2017).

[14]Since Johansen and Mertens (2016) consider the trend short-term real interest rate, we have to translate their number into the trend long-term real interest rate using the mean term spread. The mean term spread between the short-term real interest rate and the long-term real interest rate ranges from 0 percent in Ang and Bekaert (2008) to 1.1 percent in McCulloch (2009). So the long-term real rate between 1 percent and 2.6 percent is roughly consistent with the trend short-term real interest rate estimated by them.

past inflation from the nominal 10 year Treasury yield. As expected, trend estimates from the TVP-VAR model are much smoother than the ex-post measure and seem to capture low-frequency swings in the ex-post measure. Notably, tilting forecasts to match survey data evidence generally increases the median trend estimate of the long-term real interest rate than not doing so during the recent period. With tilting, fluctuations in the trend real interest rate during the recent 25 years seem to be less persistent than they are without tilting. This finding is noticeable because trend estimates with and without tilting are moderately or highly correlated for all the variables used in the TVP-VAR model, as Figures 5 $\sim$ 7 and Table 1 suggest. The fact that two median trend estimates for consumption growth are highly correlated indicates that the discrepancy in the trend long-term real interest rate is hard to explain by the different modal outlook for consumption growth. On the other hand, two median trend estimates for inflation and the nominal 10 year Treasury yield are only modestly correlated, suggesting that different outlook for these variables can explain the discrepancy in the estimated trend long-term real interest rate. The trend estimate for the nominal 10 year Treasury yield is generally much higher with tilting than without tilting, implying that the discrepancy in the estimated trend nominal interest rate might have been a big factor. The finding suggests that the future likelihood of hitting the zero lower bound on the nominal interest rate is likely to be higher when the trend follows forecasts without tilting.

3.3 Time-varying skewness in trend estimates

Recent interests in the trend long-term real interest rate were originated from the concern for secular stagnation in which trend growth and interest rate both decrease permanently. Given the zero lower bound constraint on the nominal interest rate, Summers (2016) argues that declines in both economic growth and the interest rate are likely to raise downside risks because monetary policy is less suited to address negative shocks when the economy is close to the zero lower bound. If this secular stagnation hypothesis were right, we would likely see that the downside uncertainty for consumption growth and inflation might have increased recently together with the decline in the trend long-term real interest rate. The TVP-VAR model is a useful framework to address this question because it can generate not only time-varying uncertainty but also time-varying skewness too. By looking at the skewness in trend estimates, we can judge if the downside uncertainty is more prominent than the upside uncertainty for each variable in the VAR.

As a measure of skewness, I use the following variable constructed based on various quantiles as described in Bowley (1920).

$$skew_B = \frac{Y_{q,0.25} + Y_{q,0.75} - 2Y_{q,0.5}}{Y_{q,0.75} - Y_{q,0.25}}. \tag{7}$$

$Y_{q,x}$ denotes the x-quantile of the variable Y. Colacito et al. (2016) mention that this measure and the standard moment-based skewness measure $\frac{E(Y-E(Y))^3}{V(Y-E(Y))^{1.5}}$ are propora-tional to each other. The main difference is that the quantile-based Bowley (1920) measure of skewness is bounded at $[-1, 1]$. Figures 8 $\sim$ 10 describe time-varying cross-sectional skewnesses for the variables of interest with 90% confidence bands.[15] I plot results from forecasts with and without tilting. Without tilting, consumption growth, inflation, and interest rates all show statistically significant negative skewnesses at the end of sample. In contrast, with tilting, the long-term nominal interest rate exhibits a statistically significant positive skewness at the end of sample while the long-term real interest rate shows a statitistically significant negative skewness. For other variables, skewness is not significantly different from zero at the end of sample. Overall, downside risks for consumption growth and inflation are higher when trend estimates are obtained without tilting. Regardless of tilting, the long-term real interest rate has a negative skewness near the end of sample but for different reasons. For trend estimates without tilting, the downside risk for the long-term nominal rate is a domiant factor. For trend estimates with tilting, the slight upside risk for inflation is a dominant factor. The correlation coefficients among median trend estimates in Tables 2 $\sim$ 3 suggest that relatively speaking, the long-term real interest rate is more strongly correlated with the long-term nominal interest rate without tilting while it is more strongly correlated with inflation with tilting. Therefore, the downside risk of the long-term real interest rate does not necessarily signal the high likelihood of the secular stagnation based on trend estimates with tilting.

One alternative interpretation of this finding is that survey participants can be overly optimistic about the future economy. However, for all the long-term forecasts, the uncertainty band is huge and even with tilting, the likelihood of slow growth and low inflation is non-negligible. I rather point out that trend estimates without tilting generate somewhat implausibly high chances of deflation in that the 25% quartile of trend inflation has been consistently negative since 1992. Using various specification of DSGE models with shfits to a deflation regime and the zero lower bound (ZLB) constraint, Arouba et al. (2017) find strong evidence that the U.S. has remained in a positive inflation

[15] I compute 90% confidence bands using 100 bootstrap statistics.

regime during the recent ZLB episode with a possible exception for the early part of 2009. Trend estimates with tiltiing are more consistent with their finding. Interestingly, if I use only means of survey data to tilt forecasts from the model, trend estimates still imply high chances of deflation coupled with enhanced downside risk for the long-term nominal interest rate as Figure 11 show. This finding suggests that the cross-sectional dispersion of forecasts from survey data may also provide useful external information to discipline predictive density generated from the TVP-VAR model.

4 Conclusion

This paper estimates a TVP-VAR model with stochastic volatility to obtain trend estimates for the long-term real interest rate. Rather than solely focusing on the point estimate, I generate draws from the predictive density using model estimates to account for multiple sources of uncertainty surrounding the trend. While the level and time-variation of the median long-term forecast are largely consistent with findings from the previous literature, uncertainty is much wider than typically reported in the literature when I include uncertainty about future shocks.

I tilt the original predictive density for long-term forecasts to match survey data evidence on means and variances of inflation expectations and the nominal interest rate expectations. Without tilting, the model-implied forecasts suggest a substantial downside risk for the long-run nominal interest rate and inflation since the Great recession of 2007-9, while risks are roughly balanced for these variables with tilting. The downside risk for the trend long-term real interest rate at the end of sample is similar for both forecasts but due to different reasons. With tilting, the trend long-term real interest rate is more negatively correlated with trend inflation, which shows some modest upward risk compared with estimates calculated without tilting. Without tilting, the trend long-term real interest rate is more positively correlated with the long-term nominal interest rate, which shows significant downside risk compared with estimates calculated with tilting. However, trend inflation estimates without tilting suggest implausibly high deflation risk in the U.S. since 1992, which is at odds with some sugestions in the previous literature that the risk of deflation has been modest in the U.S. even during the period involving the ZLB episode. Trend estimates with tilting to match only means of forecasts from the survey also imply implausibly high chances of deflation in the U.S. during the recent years, suggesting that cross-sectional dispersion of forecasts provides additional external information to discipline the predictive density from the TVP-VAR model.

The secular stagantion hypothesis links the permanent downward shift in the real interest rate with the predictable decline in economic growth, implying the trend inflation and the real interest rate might be stuck at the low value with narrow uncertainty bands. While the median trend estimates from the TVP-VAR model detect declines in economic growth and inflation comparable to the prediction from the secular stagnation hypothesis, the entire predicitve density of the trend is somewhat at odds with this hypothesis. In particular, the level and time variation of uncertainty of trend estimates cast some doubt on the idea that declines in trend economic growth and inflation were fairly predictable, one of the main propositions of the secular stagnation hypothesis.

Appendix

The appendix describes the details behind Bayesian estimation of the TVP-VAR model and illustrate time-varying higher order moments from the predictive density of long-term forecasts.

A.1 The elicitation of prior distributions

As explained in the main text, data from 1891 to 1914 are used to construct prior distributions for parameters and initial values of latent variables. First, I estimate the first-order VAR with constant coefficients and volatilities for the pre-sample data.

$$Y_t = \theta_0 + \theta_1' Y_{t-1} + u_t. \tag{8}$$

The prior mean of Σ_θ is obtained by shrinking the estimated covariance matrix for the VAR coefficients close to zero matrix ($0.001 \times Cov(\widehat{\theta})$). Second, $D_{ii,0}$ for $(i = 1, \cdots, N)$ is computed from the estimated variance of the ith VAR residual. Third, $\theta_{0,0}$ and $\theta_{1,0}$ are set to match the VAR coefficient estimates based on the pre-sample data. Finally, $\sigma^2_{\xi,i}$ is drawn from the inverse-gamma distribution with a single degree of freedom.

A.2 The posterior simulation

While deriving the joint posterior distribution of (ϑ, S^T) analytically is intractable, posterior distribution of a subset of (ϑ, S^T) conditional on all the rest is tractable. The exception is stochastic volatility in VAR residual shocks in which we do not have the analytical form for the conditional posterior distribution. For that part, I use Metropolis-Hastings algorithm that simulates posterior draws based on the conditional posterior kernel without knowing the exact posterior density. For other parts, I use Gibbs Sampling by taking advantage of the analytically known conditional posterior distribution. Below are some more details of the posterior simultion.

Step 1: Initialization Estimate VAR residuals (u_t) using the sample data by setting $\theta_{0,t}$ to the sample average of $Y_t - Y_{t-1}$ and $\theta_{1,t}$ to the identity matrix. Thie implicitly assumes a univariate random walk process for each variable in Y_t. Using squared VAR residuals as proxies for stochastic volatilities, one can draw $\sigma_{\xi,i}$ from the known conditional posterior distribution which is inverse gamma distribution.

Step 2: Metropolis-Hastings algorithm to draw D_{ii}^T Using the previous draw for B and VAR residuals u_t, one can construct structural shock estiamtes $\epsilon_t = Bu_t$. The variance of ϵ_t is $D_{ii,t}$. While the exact posterior distribution of $D_{ii,t}$ conditional on $(Y^T, \vartheta, \theta^T)$ is not known analytically, the conditonal posterior kernel can be calculated easily as follows:

$$p(D_{ii,t}|\vartheta, Y^T, \theta^T, D_{ii,t-1}, D_{ii,t+1}) \propto p(\epsilon_{i,t}|D_{ii,t}, \vartheta)p(D_{ii,t}|D_{ii,t-1}, \vartheta)p(D_{ii,t+1}|D_{ii,t}, \vartheta). \tag{9}$$

Under assumptions made in the TVP-VAR model, all the components in the posterior kernel are analytically known and easy to compute.

Step 3: Draw B Given D^T and u^T, off-diagonal terms in B can be computed as the regression coefficients of u_t on ε_t. For example, B is a 2×2 matrix where B_{21} is the only off-diagonal term. Because $Bu_t = \varepsilon_t$ and $B = \begin{pmatrix} 1 & 0 \\ B_{21} & 1 \end{pmatrix}$, we can run the following regressions.

$$\begin{aligned} u_{1,t} &= \varepsilon_{1,t}, \\ (D_{22,t}^{-0.5} u_{2,t}) &= B_{21}(-D_{22,t}^{-0.5} u_{1,t}) + (D_{22,t}^{-0.5} \varepsilon_{2,t}). \end{aligned} \tag{10}$$

As explained by Cogley and Sargent (2005), the above regressions imply the normal posterior for B_{21}. Generalizing this, one can draw B from the exact conditional posterior distribution which is a multivariate normal distribution.

Step 4: Draw θ^T Given ϑ, D^T, and Y^T, the TVP-VAR model is represented by a linear and Gaussian state space model for θ^T where θ_t is a latent state vector. Using

Kalman filter, one can derive the conditional posterior distribution of θ^T is normal. Carter and Kohn (1994) provides an algorithm to draw θ^T from the conditional posterior distribution by applying Kalman filter both forwards and backwards recursively.

Step 5: Draw Σ_θ Given θ^T, innovations in the evolution of VAR coefficients are known and follow normal distributions. Assuming the inverse-Wishart prior for Σ_θ, the conditional posterior distribution of Σ_θ is also inverse-Wishart and easy to simulate. Once Σ_θ is drawn, go back to **Step 1** and repeatly draw another posterior draw for ϑ and S^T.

A.3 Time-varying higher order moments in long-term forecasts: an AR(1) example

The TVP-VAR model can generate time-varying non-zero skewness as well as uncertainty for long-term forecasts. This section illustrate the property using a simple univariate process with a time-varying AR(1) coefficient as an example. Let's assume that a scalar variable Y_t follows the process below:

$$\begin{aligned} Y_t &= \theta_t Y_{t-1} + u_t \,,\; u_t \sim \mathcal{N}(0,1), \\ \theta_t &= \theta_{t-1} + \epsilon_t \,,\; \epsilon_t \sim \mathcal{N}(0,1). \end{aligned} \tag{11}$$

For simplicity, assume $\theta_t = 0$. Then $E_t(Y_{t+2}) = E_t(\epsilon_{t+1}^2)Y_t = Y_t$. The two-step ahead forecast error for Y_t can be decomposed as follows:

$$Y_{t+2} - E_t(Y_{t+2}) = Y_t(\epsilon_{t+1}^2 - 1) + (Y_t\epsilon_{t+2} + u_{t+1})\epsilon_{t+1} + u_{t+2} + u_{t+1}\epsilon_{t+2}. \tag{12}$$

Hence, the distribution of the forecast error for Y_{t+2} conditional on the information set at time t is the sum of the mixture of normal distributions and a chi-square distributions. A chi-square distribution in the express allows Y_{t+2} has a nonzero conditional skewness.

After some algebraic manipulations, one can calculate higher-order moments explicitly.

$$
\begin{aligned}
E_t(Y_{t+2} - E_t(Y_{t+2}))^2 &= 3Y_t^2 + 3, \\
E_t(Y_{t+2} - E_t(Y_{t+2}))^3 &= 14Y_t^3 + 12Y_t.
\end{aligned} \tag{13}
$$

In contrast, under the constant parameter model with $\theta = 0$.

$$
\begin{aligned}
Y_{t+2} &= u_{t+2}\,, \\
E_t(Y_{t+2}) &= 0\,, \\
E_t(Y_{t+2} - E_t(Y_{t+2}))^2 &= 1\,, \\
E_t(Y_{t+2} - E_t(Y_{t+2}))^3 &= 0.
\end{aligned} \tag{14}
$$

The above expression illustrates that the TVP-VAR model can generate complex and nonlinear dynamics of higher-order moments of the trend estimate even without stochastic volatility.

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Table 1: CORRELATION MATRIX AMONG MEDIAN ESTIMATES OF $\bar{Y}_{t,t+h+1:t+2h}$ WITH TILTING AND WITHOUT TILTING

Variable	Correlation
Δc_t	0.9026
Δd_t	0.7109
π_t	0.2471
$y_{10,t}$	0.3971
$\ln z_{d,t}$	0.9002
$r_{10,t}$	-0.0387

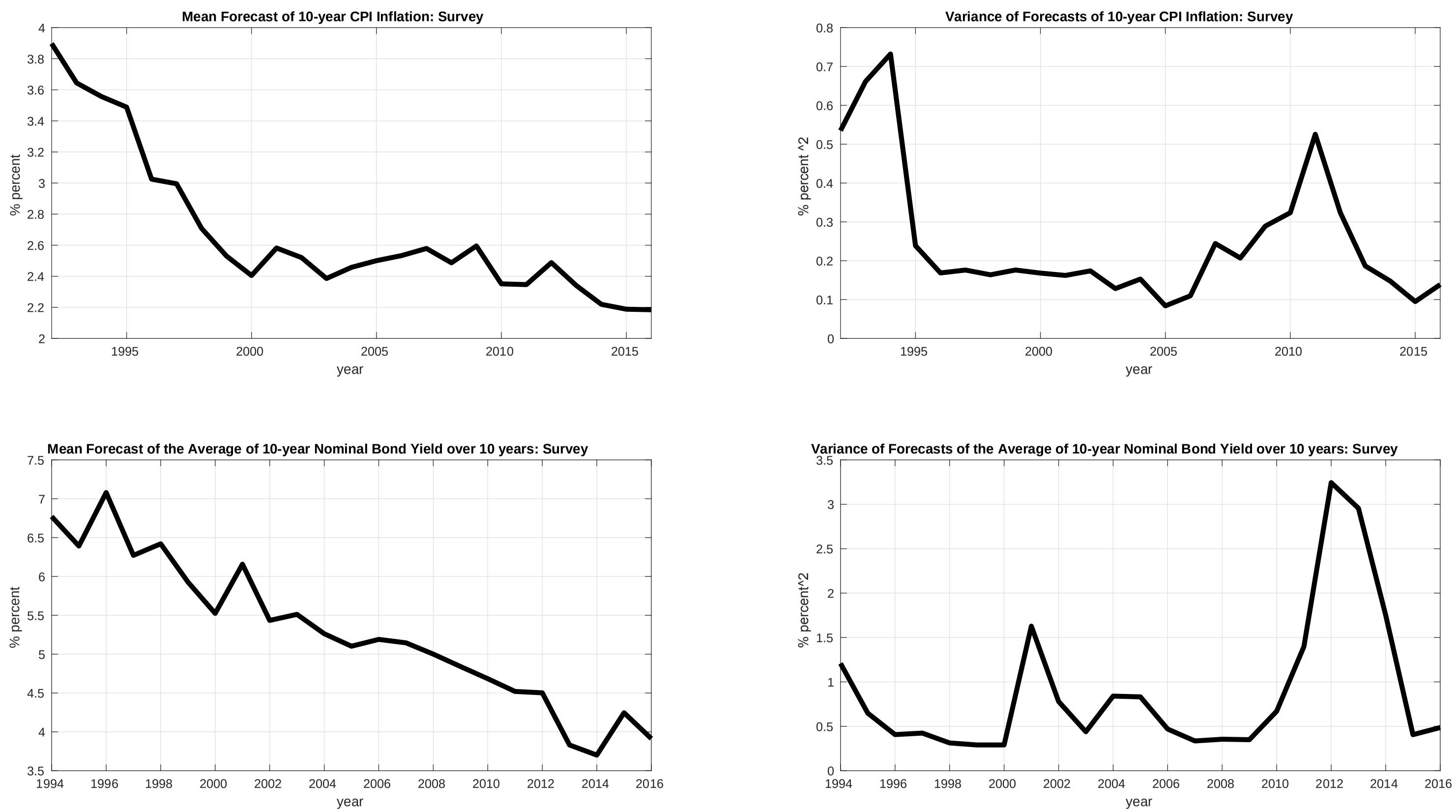
Table 2: CORRELATION MATRIX AMONG MEDIAN ESTIMATES OF $\bar{Y}_{t,t+h+1:t+2h}$ WITHOUT TILTING

Variable	Correlation
$\pi_t, y_{10,t}$	0.4894
$\pi_t, r_{10,t}$	-0.3599
$y_{10,t}, r_{10,t}$	0.6280

Table 3: CORRELATION MATRIX AMONG MEDIAN ESTIMATES OF $\bar{Y}_{t,t+h+1:t+2h}$ WITH TILTING

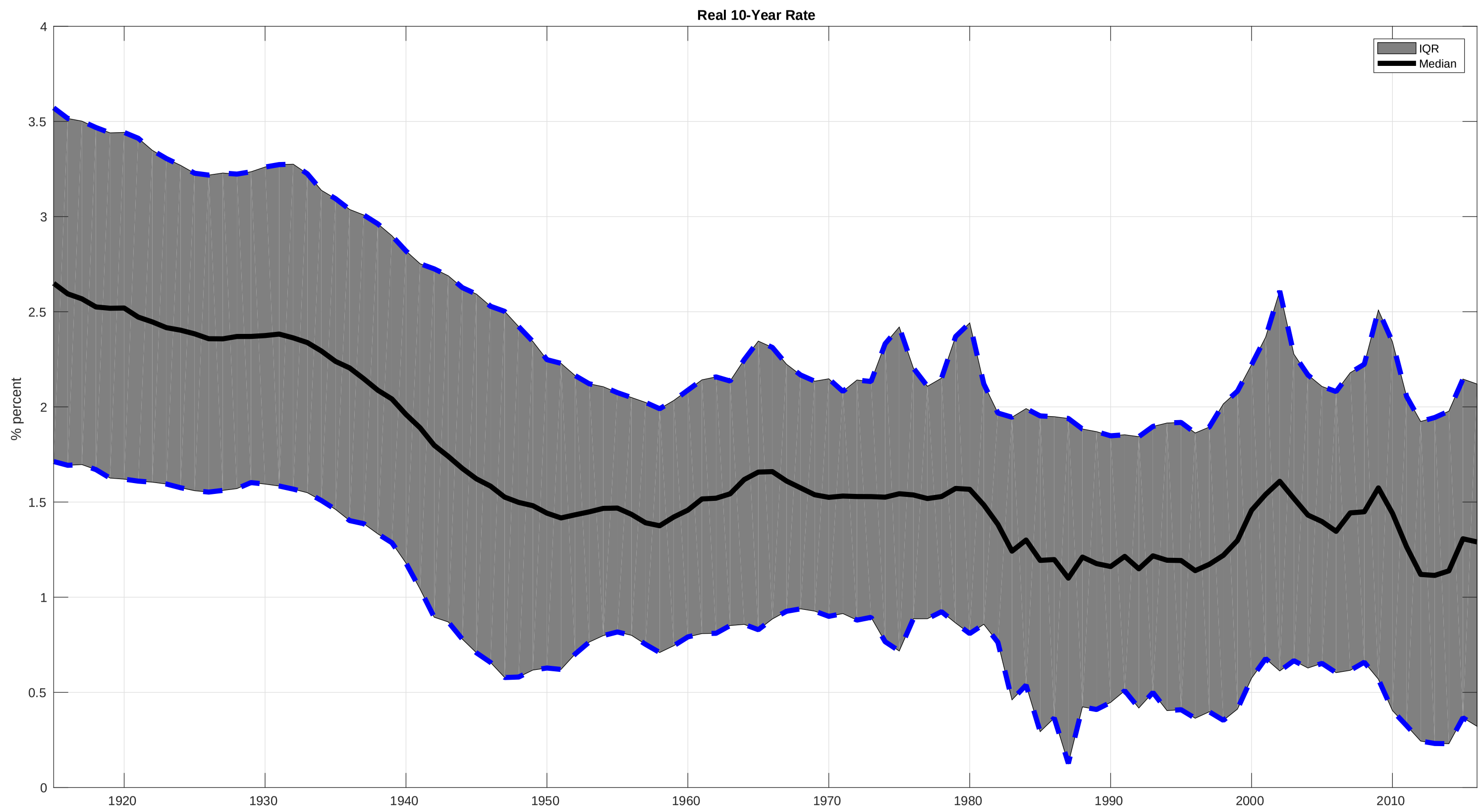
Variable	Correlation
$\pi_t, y_{10,t}$	0.4816
$\pi_t, r_{10,t}$	-0.4188
$y_{10,t}, r_{10,t}$	0.5527

Figure 1: Survey Data Evidence



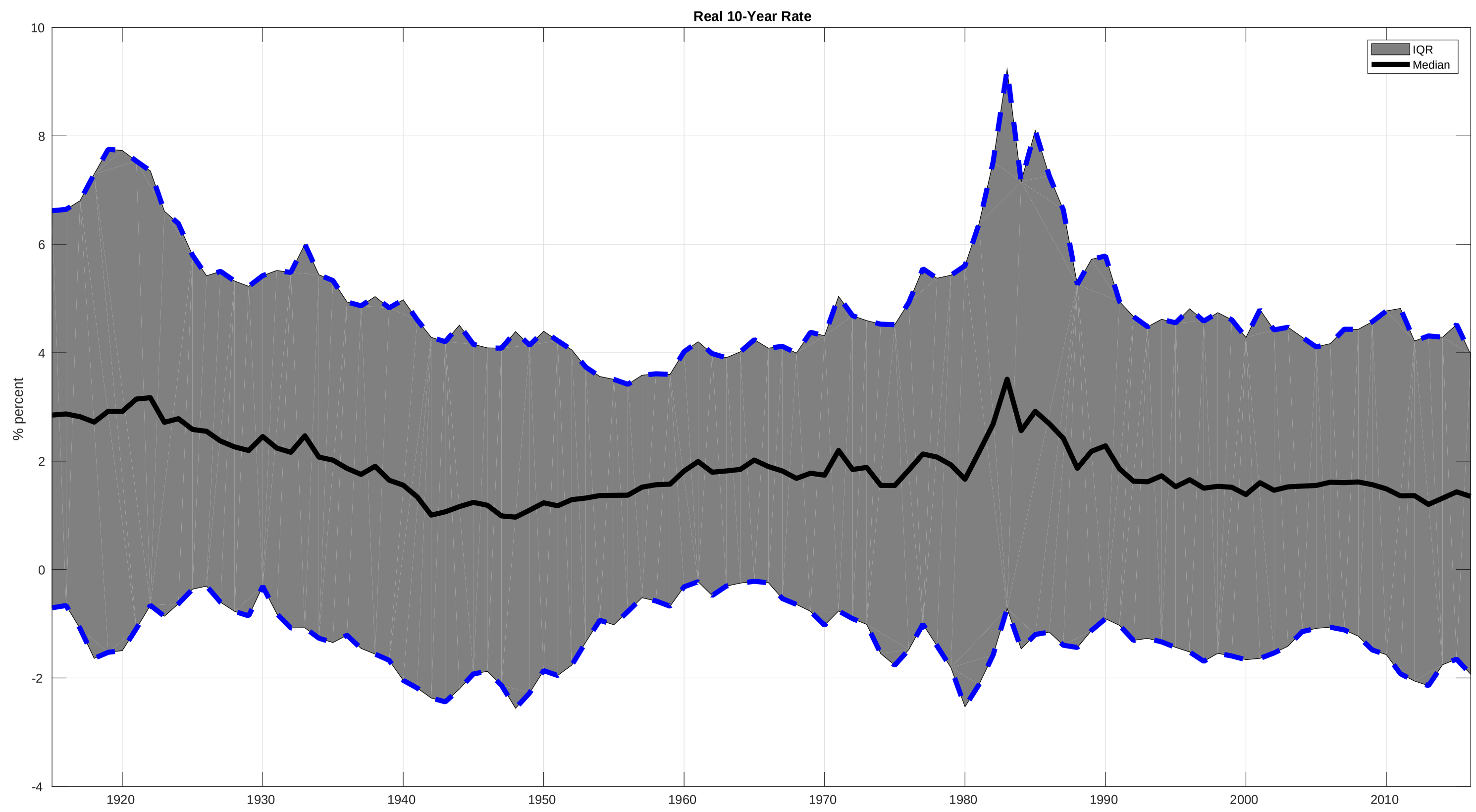
*Notes:*The underlying data are available at https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/ .

Figure 2: Trend Long-term Real Interest Rate: $\bar{Y}_t$



*Notes:*The black solid line represents the median estimate while the gray area denotes the interquartile range (IQR).

Figure 3: Trend Long-term Real Interest Rate: $\bar{Y}_{t,t+h+1:t+2h}$ without Tilting

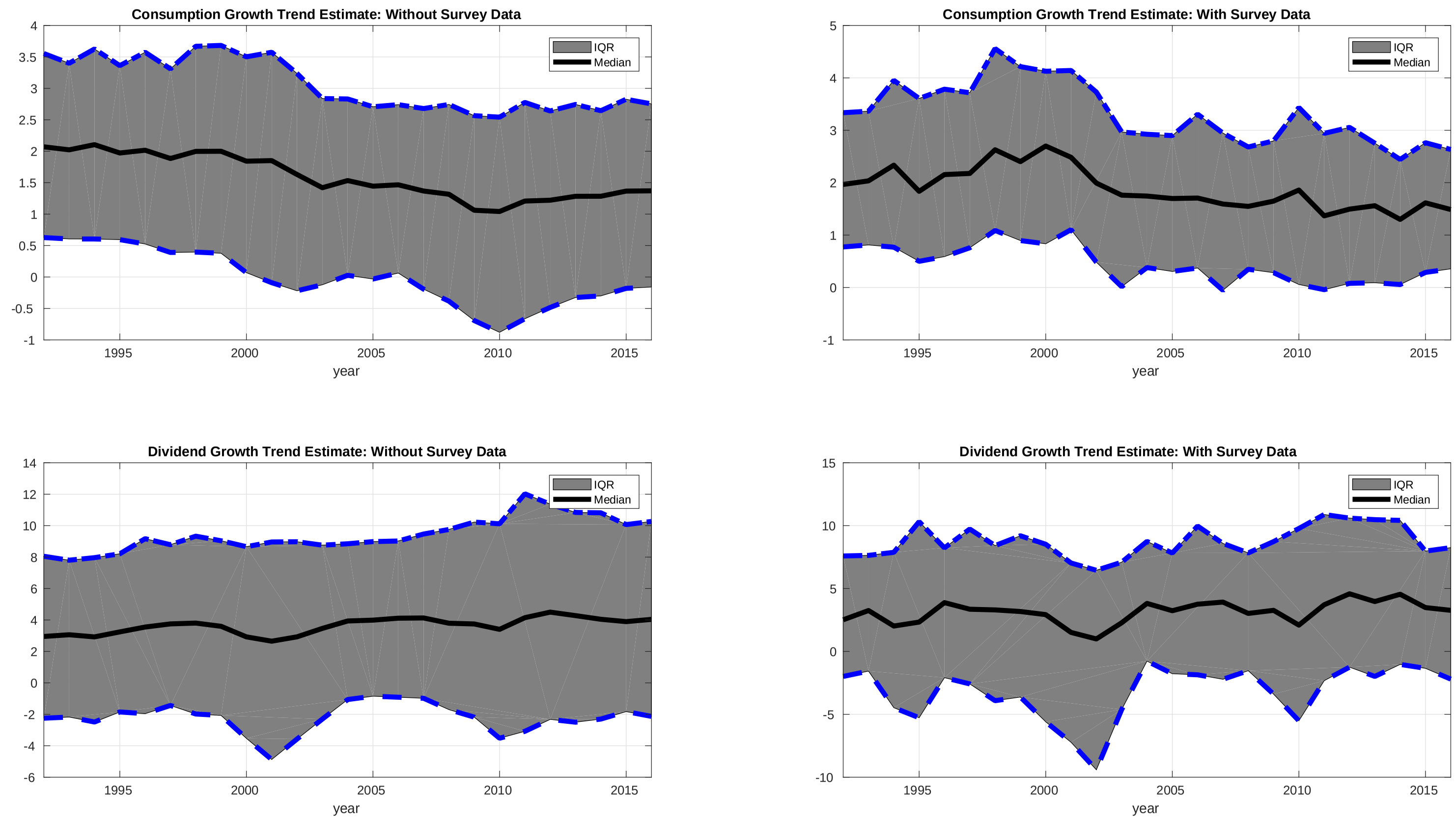


*Notes:*The black solid line represents the median estimate while the gray area denotes the interquartile range (IQR).

Figure 4: COMPARISON OF TREND LONG-TERM REAL INTEREST RATE:

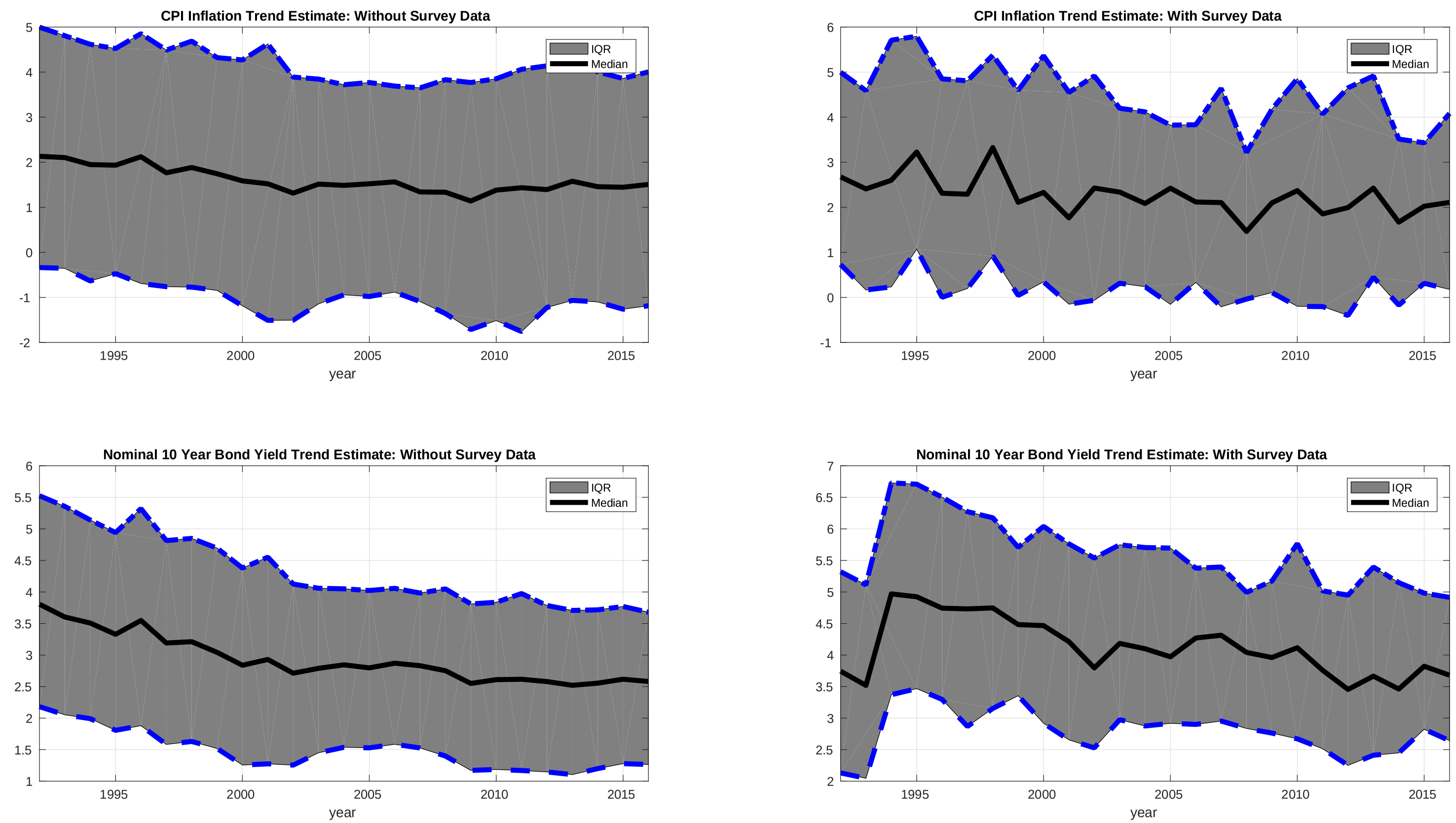
Notes: The ex-post measure of the real interest rate constructed by CEA (2015) is overlayed with two median estimates of $Y_{t,t+h+1:t+2h}$ from the TVP-VAR model and the low-frequency trend estimates using Müller and Watson (2016) projection that isolate movements of cycles longer than 16 years. For low-frequency estimates, I compute projected values for the long-term nominal interest rate and CPI inflation separately using 1891 - 2016 data and take the difference between them to obtain the trend estimate of the long-term real interest rate. Before 1992, survey data information is not avaiable and two median estimates coincide.

Figure 5: Trend Consumption and Dividend Growth: $\bar{Y}_{t,t+h+1:t+2h}$



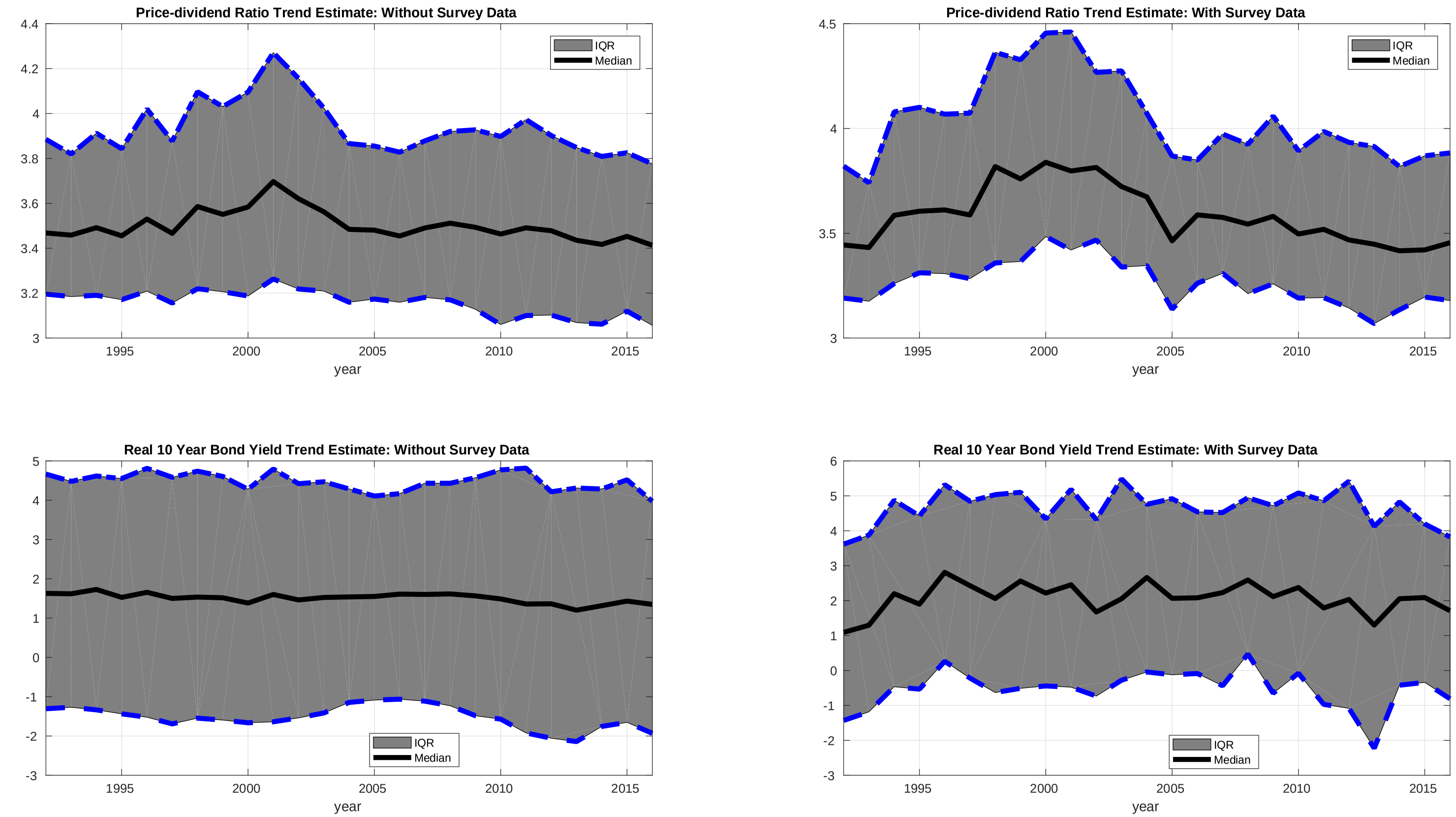
*Notes:*The black solid line represents the median estimate while the gray area denotes the interquartile range (IQR).

Figure 6: Trend Inflation and Nominal Rate: $\bar{Y}_{t,t+h+1:t+2h}$



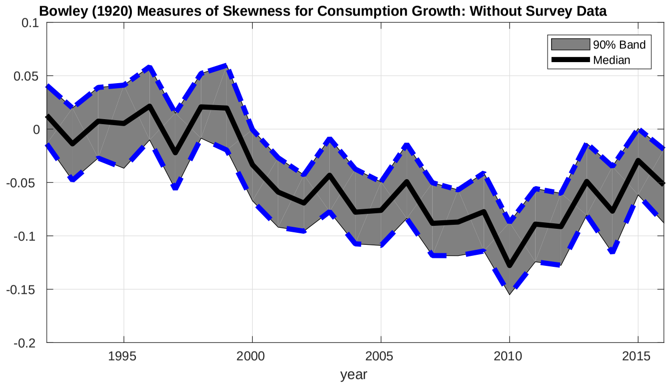
*Notes:*The black solid line represents the median estimate while the gray area denotes the interquartile range (IQR).

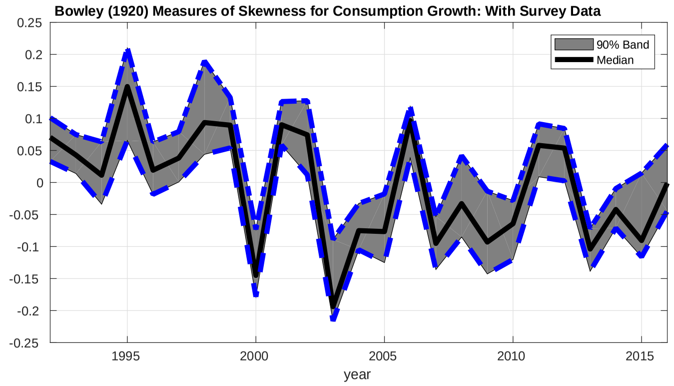
Figure 7: Trend Price-dividend Ratio and Real Rate: $\bar{Y}_{t,t+h+1:t+2h}$

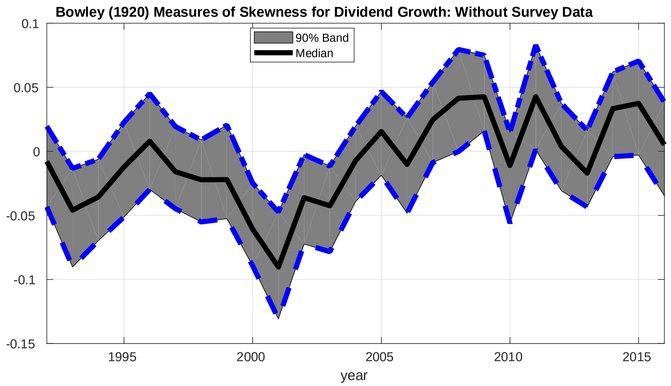


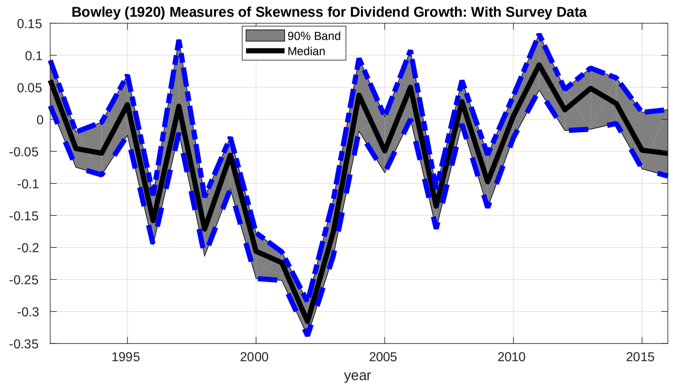
Notes: The black solid line represents the median estimate while the gray area denotes the interquartile range (IQR).

Figure 8: Time-varying Skewness: Consumption and Dividend Growth



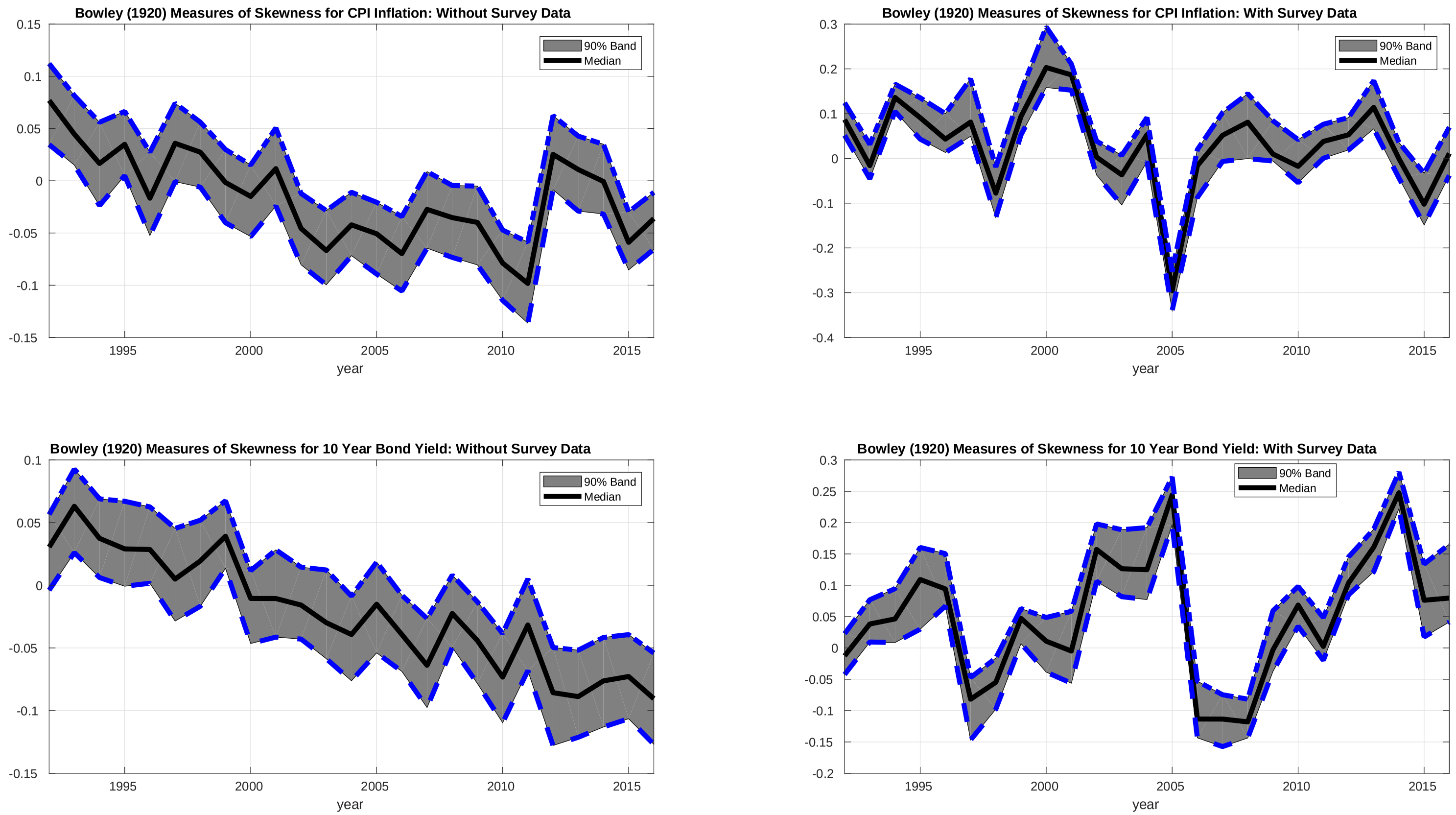






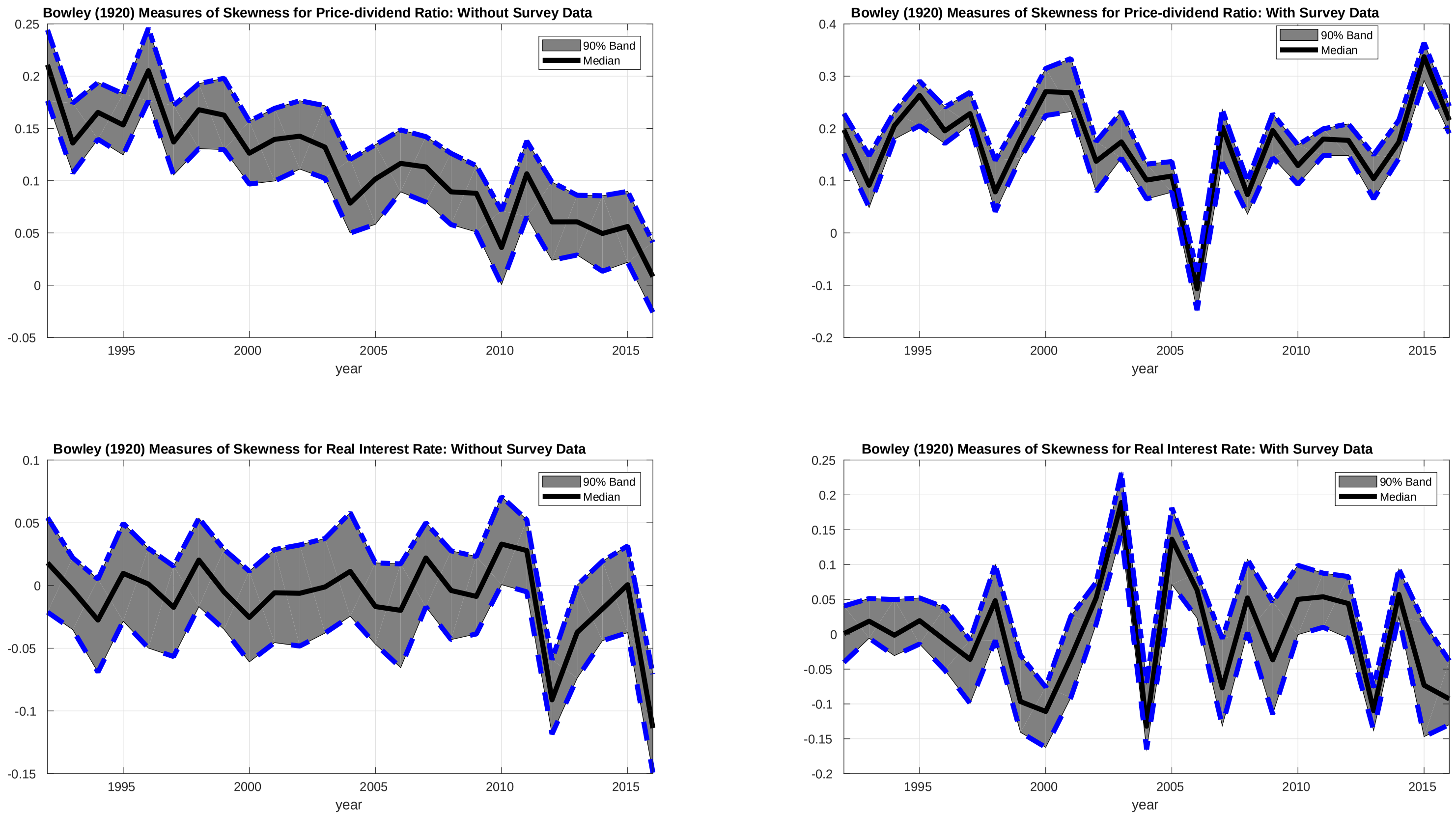
*Notes:*The black solid line represents the median estimate while the gray area denotes the 90% confidence bands from 100 statistics calculated by boostrapping.

Figure 9: Time-varying Skewness: Inflation and Nominal Rate



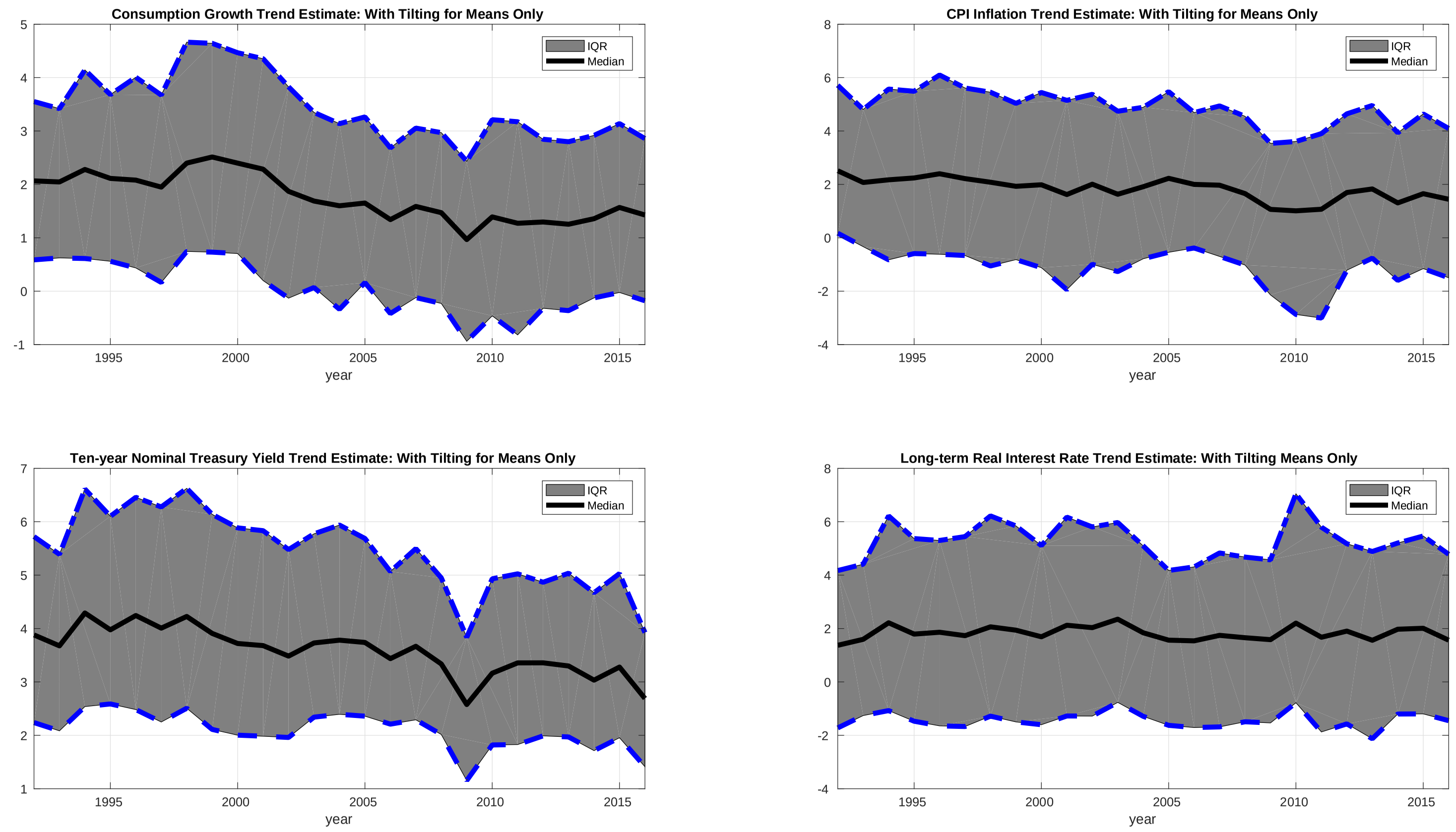
*Notes:*The black solid line represents the median estimate while the gray area denotes the 90% confidence bands from 100 statistics calculated by boostrapping.

Figure 10: Time-varying Skewness: Price-dividend Ratio and Real Rate



*Notes:*The black solid line represents the median estimate while the gray area denotes the 90% confidence bands from 100 statistics calculated by boostrapping.

Figure 11: Trend Estimates with Tilting for Means Only: $\bar{Y}_{t,t+h+1:t+2h}$



*Notes:*The black solid line represents the median estimate while the gray area denotes the interquartile range (IQR).