

# The Trend Real Interest Rate and Stagnation Risk: Bayesian Exponential Tilting with Survey Data

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# The trend real interest rate and stagnation risk: Bayesian exponential tilting with survey data

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## Abstract

The decline in the real interest rate during the recent few decades coupled with the Great Recession of 2007-2009 raised a concern that the U.S. economy might face stagnation like the Japanese economy since the late 1990s. The increased likelihood of the zero lower bound (ZLB) on the nominal interest rate that constrains the effectiveness of monetary policy at the low equilibrium real interest rate is often cited as a cause of stagnation. However, the central bank's unconventional policies such as large scale asset purchases and forward guidance can mitigate stagnation risk arising from the ZLB constraint. To empirically assess the impact of these opposing forces on the risk of stagnation, this paper uses long-horizon predictive distributions of macro variables from a time-varying parameter vector autoregression (TVP-VAR) model. While the concern for long-term (five-year ahead) stagnation risk due to the ZLB constraint on monetary policy appears to be justified from the purely model-based predictive distributions for macro variables, the risk substantially declines when these predictive distributions are tilted to match both cross-sectional means and variances from survey forecasts of inflation

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and the nominal interest rate. And the probability score ranking based on the prediction accuracy of downside tail events favors the predictive distribution with tilting. This finding affirms the view that unconventional monetary policies as conducted by the Federal Reserve were effective in reducing stagnation risk by influencing the private sector's expectations.

**Key Words:** Trend real interest rate; Stagnation risk; Time-varying parameter vectorautoregression; Bayesian estimation.

**JEL Classification:** C11, E43

# 1 Introduction

Many empirical studies document that the trend component of the real interest rate declined over the last few decades in the U.S. (Christensen and Rudebusch (2019); Council of Economic Advisers (2015); Del Negro et al. (2017); Hamilton et al. (2016); Holston et al. (2017); Johannsen and Mertens (2016); Kiley and Roberts (2017); Lewis and Vazquez-Grande (2018); Lunsford and West (2018); Rachel and Smith (2015), etc.). While the precise estimate of the decline in the trend real interest rate varies across different studies, many researchers agree that the current level of the equilibrium real interest rate that would prevail over the long-run is most likely to be closer to 1 percent after falling by about 2 percentage points since the mid 1980s.(Christensen and Rudebusch (2019); Hamilton et al. (2016), for example).<sup>1</sup> In addition, the median projection of the longer-run real federal funds rate by the FOMC participants also has declined to 0.8 percent from 2.3 percent over the last seven years assuming that inflation expectations are anchored at 2 percent target (Clarida (2019)).

The decline in the trend real interest rate raised a concern that it might be a signal that the economy entered a “new normal” or the “secular stagnation” that would be characterized by low growth and low inflation with the increased likelihood of being stuck at the zero lower bound (ZLB) on the nominal interest rate (Eggertsson et al. (2019); Fischer (2016); Gagnon et al. (2016); Rachel and Summers (2019); Summers (2016)). The secular stagnation hypothesis was originally proposed by Hansen (1939) during the inter-war period to explain the persistent underemployment and the low level of investment activities in spite of the low interest rate level. Hansen (1939) attributed slow growth to the lack of aggregate demand due to declining population growth. Similarly, current research on the secular stagnation links the downward shift in the real interest rate with the decline in growth and inflation through demographic changes as shown in the Japanese

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<sup>1</sup>The trend real interest rate estimated from statistical models can be an empirical proxy for the equilibrium real interest rate in more structural models because temporary shocks creating the wedge between the equilibrium real interest rate and the actual real interest rate would dissipate over the long-run. For a similar reason, the trend long-term real interest rate and the trend short-term real interest rate are assumed to move closely in this paper because long-horizon expectations of the short-term real interest rate would be a major determinant of both measures.

experience since the late 1990s. For example, Gagnon et al. (2016) argue that a substantial portion of the decline in both real GDP growth and the equilibrium real interest rate since 1980 could be fairly predictable because these declines were driven by the aging of the post-WWII baby-boom generation. Similarly, Eggertsson et al. (2019) show that the decline in the equilibrium real interest rate due to slowly-moving demographic and technological factors can exacerbate negative tail risks for growth and inflation when combined with the ZLB constraint on the nominal interest rate and the central bank targeting low inflation. Rachel and Summers (2019) argue that the risk of stagnation was masked only by expansionary fiscal policies such as a higher government debt relative to GDP. However, this type of analysis is mostly based on the exercise using calibrated models and focuses on point estimates, and thus, may underestimate the uncertainty of assessing the probability of tail events.<sup>2</sup>

This paper considers a time-varying parameter vectorautoregression (TVP-VAR) model that can take into account multiple sources of uncertainties residing in the long-run projection of macroeconomic variables. In particular, the model considers the uncertainty from low-frequency variations in the relationship among consumption growth, inflation, and asset prices as well as the uncertainty from temporary shocks in making long-run projections for macro variables and assessing tail risks. In addition, I adopt the Bayesian framework to estimate the TVP-VAR model as in Cogley and Sargent (2005) and Cogley, Morozov, and Sargent (2005) and include parameter uncertainty in the evaluation of risks. I use simulated predictive distributions of observed variables from the TVP-VAR model to estimate the trend real interest rate and evaluate stagnation risk.

Using simulated predictive distributions has three main advantages in estimating the trend components of VAR variables and tail risks. First, long-horizon predictive distributions from the TVP-VAR model can exhibit non-normal properties such as time-varying non-zero skewnesses and fat tails even if all the underlying

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<sup>2</sup>Kiley and Roberts (2017) calculate the likelihood of hitting the ZLB at different values of the equilibrium real interest rate using a large-scale macro model (FRBUS) used by the staff of the Federal Reserve Board and show that chances become higher as the equilibrium interest rate gets lower. However, they do not assess the probability that the equilibrium real interest rate itself takes a particular value and does not provide empirical estimates of tail risks.

shocks are following symmetric normal distributions. This property is present because shocks to VAR coefficients have multiplicative effects on observed variables and, therefore, generate nonlinear impacts. Second, predictive distributions from the model can be tilted to match information outside the model. Such a tilting is found to improve the accuracy of the model-based prediction (Cogley, Morozov, and Sargent (2005); Krüger et al. (2017); Robertson, Tallman, and Whiteman (2005); Tallman and Zaman (2018)). Third, simulated predictive distributions can include uncertainty about future shocks to VAR coefficients which can have nonlinear effects on VAR variables. In contrast, approximating the trend as the model-implied unconditional mean in the absence of future shocks to VAR coefficients as done in Cogley and Sargent (2005) underestimates the uncertainty about the trend component.

As pointed out by Lewis and Vazquez-Grande (2018), different priors for the time variation in latent variables can generate substantially different posterior distributions in Bayesian methods, and I consider both tight and loose priors for parameters determining the magnitude of the time variation in VAR coefficients. I take the one with loose priors as the baseline specification and treat the one with tight priors as an alternative specification. Then, I rank different predictive distributions using the probability scoring metric in terms of their accuracy of predicting downside tail events (Gneiting and Ranjan (2011)).

The main findings from the empirical analysis using the long-run historical data in the U.S. can be summarized as follows. First, the decline in the median estimate of the trend real interest rate since the mid 1980s from the TVP-VAR model is about 2.6 percent, in line with findings from the existing literature. However, the uncertainty band is at least twice as big as the one in Holston et al. (2017). The higher uncertainty in the TVP-VAR model appears to be driven by future shock uncertainty because the unconditional mean measure ignoring future shock uncertainty shows an uncertainty band comparable to the one in Holston et al. (2017). According to estimates from the TVP-VAR model, the trend real interest rate did not show a sudden drop during the Great Recession and its decline since the mid 1980s was very gradual. Despite this difference, the median estimate of the trend real interest rate is highly correlated with various measures from the

existing literature. Second, the risk of stagnation increased suddenly during the Great Recession compared to the post-WWII average level though it declined subsequently. However, the long-term (five-year ahead) risk of stagnation declined more sluggishly than the near-term (one-year ahead) risk of stagnation. Third, when predictive distributions are tilted to match both cross-sectional means and variances of survey forecasts for inflation and the nominal interest rate, the risk of stagnation became generally lower and declined more rapidly even for the long-term. Interestingly, tilting only cross-sectional means of survey forecasts does not generate a comparable reduction in stagnation risk. The probability scoring rule generally favors the predictive distribution with tilting and loose priors for the time-variation in VAR coefficients. The result supports the view that the Federal Reserve's unconventional policies were effective in reducing stagnation risk by influencing the private sector's expectations.

These findings on stagnation risk are less pessimistic about the constraint on monetary policy than Eggertsson et al. (2019) and Rachel and Summers (2019), who emphasize the role of fiscal policies and doubt the effectiveness of monetary policy at the ZLB in reducing stagnation risk. Stagnation risk abated during 2013-2014 when fiscal policy stance was mostly contractionary while monetary policy remained accommodative. However, the conclusion in this paper is in line with Aruoba et al. (2018) who suggest that the aggressive policy responses of the Federal Reserve might have succeeded in coordinating inflation expectations near its target and reducing the tail risk based on an estimated New Keynesian model subject to the ZLB. They contrast this result with the Japanese experience during the late 1990s, for which they find the actual transition to a deflation regime. Similarly, Debortoli et al. (2018) argue that the Federal Reserve's unconventional policies were largely successful in getting around the constraint on conventional monetary policy due to the ZLB. Their analysis is based on a time-varying parameter structural vectorautoregression (TVP-SVAR) model showing that inflation, labor productivity and hours worked respond similarly to a measure of monetary policy shock (an innovation to long-term nominal bond yields) before and after the ZLB period. Hattori et al. (2016) also find that the announcements of unconventional monetary policies substantially reduced option-implied equity market tail

risks. In sum, although the decline in the equilibrium real interest rate and the ZLB constraint on monetary policy increased stagnation risk in the aftermath of the Great Recession, the Federal Reserve’s aggressive use of unconventional monetary policies played a significant role in reducing stagnation risk based on both macro and financial market data.

This paper proceeds as follows. Section 2 introduces the TVP-VAR model estimated in this paper and describes the data used. Section 3 describes how to construct a predictive density for trend using long-horizon forecasts from the TVP-VAR model with tilting. Section 4 describes empirical analysis of the trend real interest rate and stagnation risk using the U.S. data. Section 5 concludes.

## 2 A Time-varying Parameter Vectorautoregression Model

The TVP-VAR model used in this paper includes five observed variables: real per-capita consumption growth ( $\Delta c_t$ ), real per-capita dividend growth ( $\Delta d_t$ ), CPI inflation ( $\pi_t$ ), the 10 year nominal interest rate ( $y_{10,t}$ ), and the log of the aggregate price-dividend ratio ( $\ln z_{d,t}$ ). Not only are VAR coefficients in the model time varying, but each variable in the model is also subject to a shock with time-varying volatility. I describe the details on the construction of the data and the structure of the TVP-VAR model below.

### 2.1 Data

The estimation uses long-run annual observations for the U.S. from 1891 and 2016. Observations from 1891 to 1918 are used only to construct prior distributions. For consumption growth from 1930 to 2016, I use the year-over-year growth rate of per-capita real consumption expenditures available from Haver Analytics while the earlier data draw on Kendrick’s measures based on real consumption of non-

durable goods and services.<sup>3</sup> The 2016 observation for consumption growth is the growth rate of consumption from year 2014 to year 2015. This is done to match the information set contained in interest rates and stock prices that are based on the January values of the corresponding year. For dividend growth, I use the December-to-December growth rate in the previous year. For instance, 1992 dividend growth is calculated as the growth from December 1990 to December 1991. Inflation is defined by the change in the price level from the January of the previous year to the January of the current year while price-dividend ratio and the 10-year Treasury yield are the January level of the corresponding year. Given the low frequency of data, I estimate a first-order VAR model.

I include  $\Delta d_t$  and  $\ln z_{d,t}$  because stock market data provide information about the expected future real discount rate that moves with the long-term real interest rate.<sup>4</sup> Under the Gordon growth model in which investors expect dividend growth and discount rate in the future to be unchanged ( $E_t(\Delta d_{t+j}) = \Delta d_t$ ,  $E_t(r_{s,t+j}) = r_{s,t}$ ), the real discount rate for stocks ( $r_{s,t}$ ) can be backed out from the price-dividend ratio and dividend growth rate using the following formula:

$$z_{d,t} = \frac{P_t}{D_t} = \sum_{j=1}^{\infty} \frac{\left(\frac{D_{t+s}}{D_t}\right)}{(1+r_{s,t})^j} \rightarrow z_{d,t} = \frac{1 + \Delta d_t}{r_{s,t} - \Delta d_t} \rightarrow r_{s,t} = \Delta d_t + \frac{1 + \Delta d_t}{z_{d,t}}. \quad (1)$$

If we plug the price-dividend ratio and dividend growth rate at each period, we can back out the time-series of the implied real discount rate for stocks.

Figure 1 shows variables used in the VAR model along with the implied real discount rate for stocks. Consumption growth and inflation became more stabilized during the post-WWII period relative to the pre-WWII period while asset prices such as the 10 year nominal interest rate showed more persistent and volatile

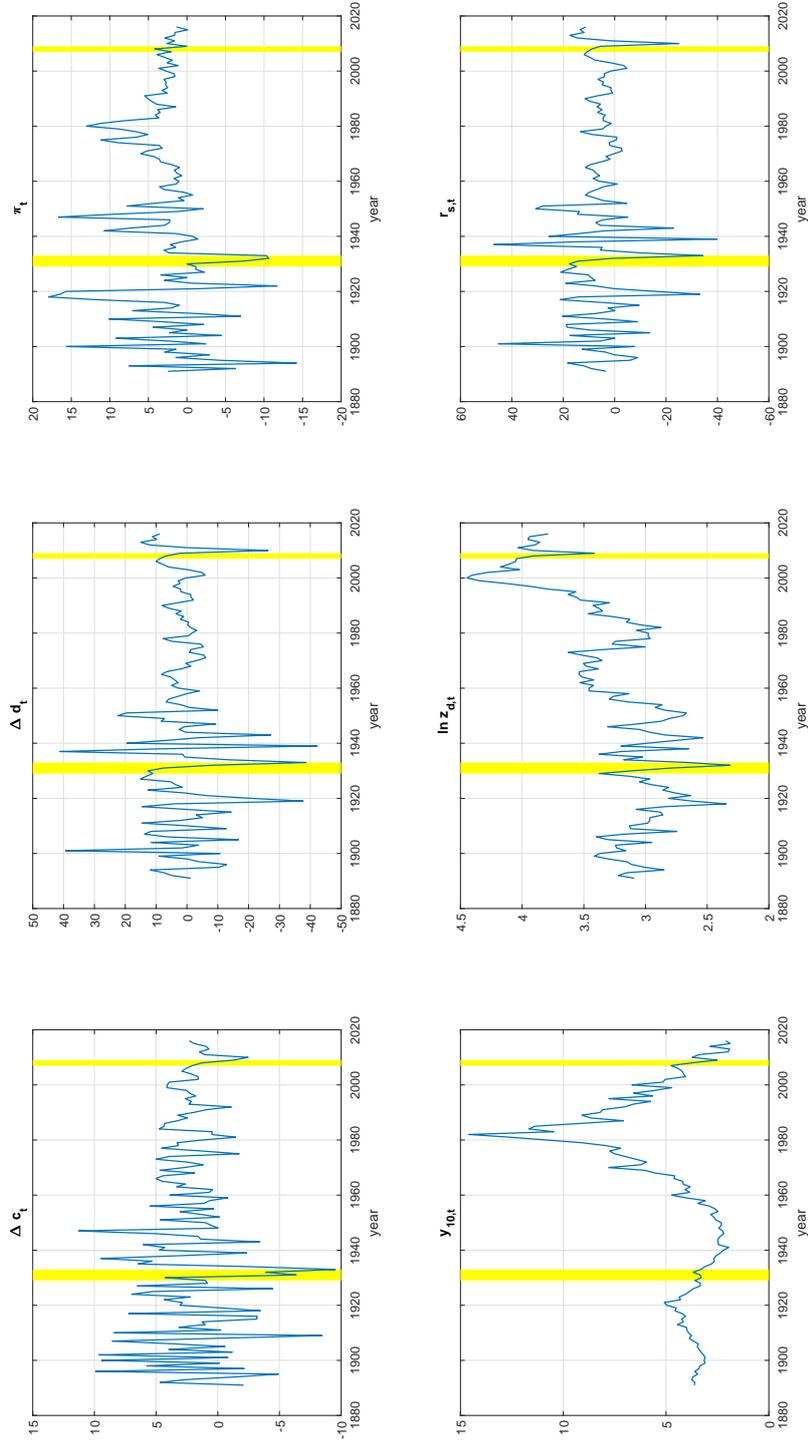
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<sup>3</sup>Except for consumption growth after 1930, all the other observations are obtained from Robert Shiller's website ([www.econ.yale.edu/shiller/data.htm](http://www.econ.yale.edu/shiller/data.htm)).

<sup>4</sup>Under Campbell-Shiller (1988) log-linear approximation,  $\ln z_{d,t} \approx \frac{\kappa}{1-\rho} + \sum_{j=0}^{\infty} \rho^j E_t(\Delta d_{t+1+j} - r_{s,t+1+j})$  where  $r_{s,t+1+j}$  is the real discount rate at  $t+1+j$ , which is the sum of the real interest rate and the risk premium, where  $\rho$  and  $\kappa$  are constants determined by the steady-state log price-dividend ratio.

movements during the post-WWII period. The Great recession of 2007-09 is often compared to the Great Depression of 1929-33, but the magnitude of the decline in growth and inflation was far severe during the Depression although the decline in the long-term bond yield and real discount rate for stocks was comparable. Given the fact that the Federal Reserve targeted the long-term bond yield at the ZLB on the short-term nominal interest rate between December 2008 and December 2015, this observation suggests that monetary policy responses relative to the size of negative shocks hitting the economy might have been more aggressive during the Great Recession period than the Great Depression period.

Figure 1: DATA



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*Notes:* The implied real discount rate for stocks is calculated using the formula explained in the text. Shaded areas include the Great Depression (1929-1933) and the Great Recession (2007-2009) periods.

Forecasts for CPI inflation and the 10-year Treasury yield from the Survey of Professional Forecasters (SPF) and 1-year forward CPI inflation 1 year from now (1yr1yr forward) from Blue Chip economic indicators are used to tilt the distribution of model-implied forecasts toward observed proxies for expectations.<sup>5</sup> Survey data of average inflation expectations over ten years are available from 1992 while the expectations of the 10-year Treasury yield averaged over ten years are available from 1994. Blue Chip forecasts for 1-year ahead 1-year forward CPI inflation are available from 1985. Figure 2 describes cross-sectional means and variances of available survey forecasts. Mean forecasts for inflation and the nominal interest rate gradually declined but cross-sectional variances fluctuated over the period. While the cross-sectional dispersion of short-term inflation expectations rose during the Great Recession but fell quickly when the recession ended, the cross-sectional dispersion of long-term inflation expectations and the long-term bond yield continued to rise even after the recession ended and started to decline later.

## 2.2 A TVP-VAR model with stochastic volatility

I denote the vector of five observed variables by  $Y_t = [\Delta c_t, \Delta d_t, \pi_t, y_{10,t}, \ln z_{d,t}]'$ . The dynamics of  $Y_t$  in the TVP-VAR model can be described as follows:

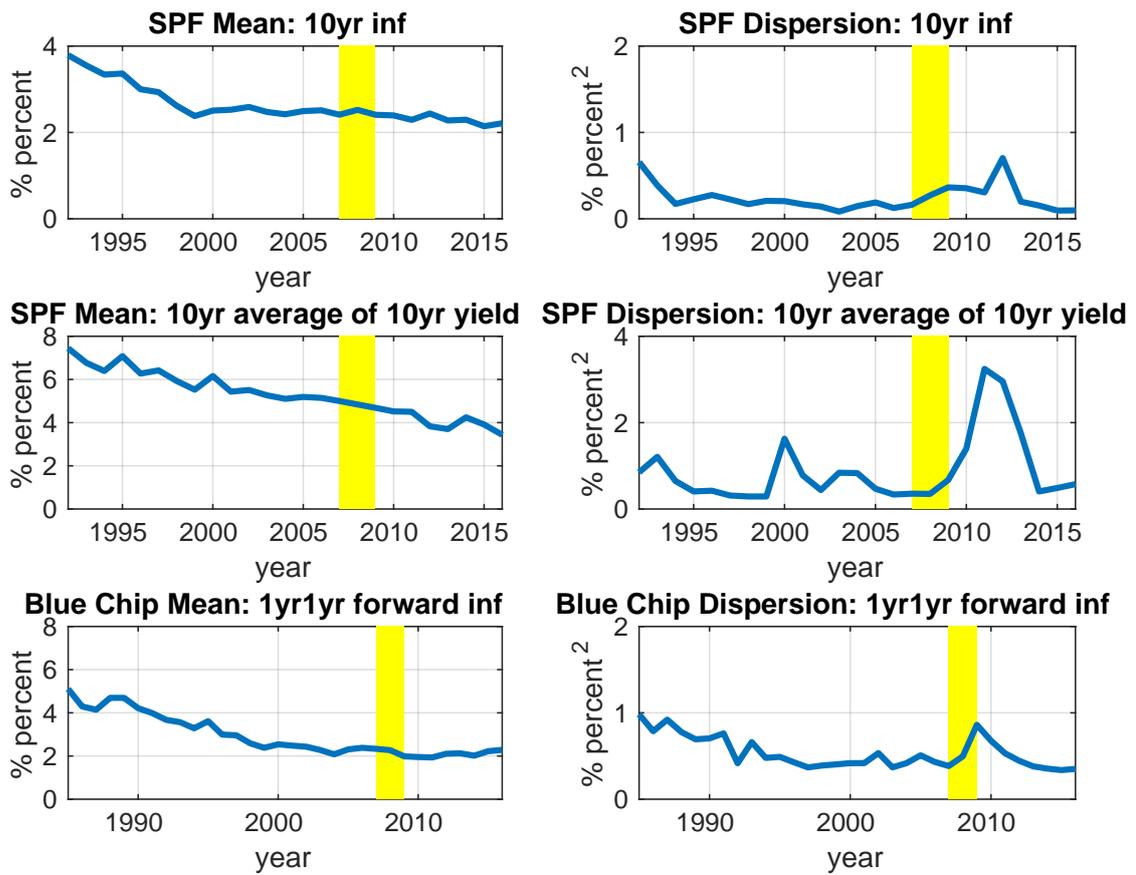
$$\begin{aligned}
Y_t &= \theta_{0,t} + \theta_{1,t}Y_{t-1} + u_t, \quad u_t = B^{-1}\varepsilon_t \sim \mathcal{N}(0, \Sigma_t), \\
\theta_{0,i,t} &= \theta_{0,i,t-1} + \epsilon_{\theta_{0,i,t}}, \quad (i = 1, \dots, 5), \\
\theta_{1,i,j,t} &= \theta_{1,i,j,t-1} + \epsilon_{\theta_{1,i,j,t}}, \quad (i, j = 1, \dots, 5), \\
\Sigma_t &= (B^{-1})'D_t(B^{-1}), \quad \ln D_{ii,t} = \ln D_{ii,t-1} + \sigma_{\xi,i}\xi_{i,t}, \quad \xi_{i,t} \sim \mathcal{N}(0, 1), \quad (i = 1, \dots, 5), \\
\theta_t &= \text{vec}([\theta_{0,t}, \theta_{1,t}]), \quad \epsilon_{\theta,t} = \text{vec}([\epsilon_{\theta_{0,t}}, \epsilon_{\theta_{1,t}}]), \quad \Sigma_\theta = E(\epsilon_{\theta,t}\epsilon_{\theta,t}'), \quad \epsilon_{\theta,t} \sim \mathcal{N}(0, \Sigma_\theta).
\end{aligned} \tag{2}$$

where  $D_t$  is a diagonal matrix. The model estimates consist of two parts: the first part is the collection of constant parameters ( $\vartheta = [B, \Sigma_\theta, \sigma_{\xi,i}]$ ) and the second part is the collection of latent variables including time-varying coefficients and

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<sup>5</sup>For SPF, observations are taken from the survey during the first quarter of the year. For Blue Chip economic indicators, the January survey is used.

Figure 2: SURVEY DATA



Notes: The Great Recession (2007-2009) period is shaded.

time-varying volatilities ( $S_t = [\theta_t, D_t]$ ).

The Bayesian estimation approach adopted in this paper first constructs the prior distribution for  $\vartheta$  and the initial estimates of latent variables  $S_0$  ( $p(\vartheta, S_0)$ ). I use the pre-sample data from 1891 to 1914 to estimate a constant parameter VAR with constant volatility. The prior distributions for  $S_0$  are obtained using the estimates of VAR coefficients and residuals from these pre-sample estimation results. The prior for  $\Sigma_\theta$  that determines the amount of variation in time-varying coefficients is important for controlling the variation of trend estimates. Following Primiceri (2005) and Del Negro et al. (2017), I set up the prior mean of  $\Sigma_\theta$  so that the implied expected change in trend estimates during the pre-sample period is reasonable. I consider two priors for  $\Sigma_\theta$  in which the prior mean of  $\Sigma_\theta$  in the loose prior is four times bigger than that in the tight prior. The tight prior constrains the amount of time-variation in trend to make the implied expected change in the trend component less than or equal to the one-standard-deviation of each variable in  $Y_t$  while the loose prior relaxes this restrictions.<sup>6</sup>

Under the Bayesian approach, prior information is combined with the sample data to obtain the posterior distribution. In the constant parameter VAR model with constant volatility, we can compute the likelihood  $p(Y^T|\vartheta)$  analytically and combine this information with conjugate priors to derive the posterior distribution of  $p(\vartheta|Y^T)$  under the normality assumption of shocks.<sup>7</sup> Due to the fact that time-varying coefficients in the VAR make variables in the VAR nonlinear functions of the past shocks, it is not feasible to easily compute the likelihood function for observed variables in the TVP-VAR, which integrates out unobserved latent state variables. In addition, time-varying volatilities generate another type of nonlinearities. Following Cogley and Sargent (2005), I rely on the posterior simulation of  $\vartheta$  and  $S^T$  using the property that the distribution of each subset in  $(\vartheta, \theta_t, D_t)$  conditional on the rest of the model is easy to simulate. To exclude parameters implying the explosive dynamics, I remove  $\vartheta$  and  $S^T$  implying that  $Y_t$  does not follow a stationary process at any point of time. Out of 100,000 posterior draws of  $(\vartheta, S^T)$ , the first 50,000 draws are burned in and the remaining 50,000 draws

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<sup>6</sup>The other details of prior elicitation and posterior simulation are explained in the appendix.

<sup>7</sup> $Y^T$  stands for  $[Y_1, \dots, Y_T]$ .

are used for analysis.<sup>8</sup>

### 3 Predictive Density for Trend

This section consists of four parts. First, I describe different definitions of trend estimates of  $Y_t$  and examine sources of uncertainties surrounding their predictive distributions. Second, nonlinear and non-normal properties of predictive distributions are explained. Third, I explain how to tilt predictive densities to match first and second moments from the survey data. Finally, the method to score predictive distributions based on their accuracy to forecast negative tail events is explained.

#### 3.1 Sources of uncertainty in predictive distributions of $Y_t$

As illustrated by Cogley, Morozov, and Sargent (2005), the TVP-VAR model with time-varying volatility exhibits multiple uncertainties for long-horizon forecasts. Denote forecasts for  $[Y_{t+1}, \dots, Y_{t+h}]$  by  $Y_{t+1}^{t+h}$ . There are three different types of uncertainty behind the predictive density  $p(Y_{t+1}^{t+h}|Y^t)$ .

$$\begin{aligned} p(Y_{t+1}^{t+h}|Y^t) &= \underbrace{p(\vartheta, S^t|Y^t)}_{\text{(parameter)}} \times \underbrace{p(S_{t+1}^{t+h}|\vartheta, S^t)}_{\text{(VAR coefficients and volatility)}} \times \underbrace{p(Y_{t+1}^{t+h}|\vartheta, Y^t, S_{t+1}^{t+h})}_{\text{(VAR residual shocks)}}, \end{aligned} \quad (3)$$

Following Amir-Ahmadi, Matthes, and Wang (2016), we can define the trend component as a long-horizon ( $h$ -step ahead) forecast. To better isolate low frequency movements in long horizon forecasts, I adopt the following average of long-horizon forecasts as a measure of the trend:<sup>9</sup>

$$\bar{Y}_{t,t+h+1:t+2h} = \frac{\sum_{j=1}^h (Y_{t+h+j})}{h}. \quad (4)$$

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<sup>8</sup>For actual inference, I only save every 10th draw. Therefore, 5,000 draws are used for posterior inference discussed below.

<sup>9</sup>Müller and Watson (2017) measure uncertainty about long-run predictions using the similar long-horizon average of forecasts. I again impose the stationarity restriction on parameter draws and consider long-term forecasts from posterior draws satisfying the stationarity condition.

The predictive density of  $Y_{t,t+h+1:t+2h}$  ( $p(Y_{t,t+h+1:t+2h}|Y^t)$ ) can incorporate future shock uncertainty as well as parameter uncertainty. For  $h$  large enough, such as 10 years, this measure can be a good proxy for the long-term trend. I compute the ex-ante real interest rate ( $r_{10,t}$ ) by subtracting the next 10-year average inflation forecast from the forecast for the 10-year nominal interest rate. One can calculate the predictive density by simulating  $Y_{t+h+j}$  from each posterior draw of  $\vartheta$  and  $S^t$ . This predictive density can consider future shock uncertainty as well as parameter uncertainty.

One popular alternative concept of the trend in the literature is to take the model-implied long-run mean of  $Y_t$  conditional on no future shocks to time-varying coefficients.

$$\bar{Y}_t = \lim_{h \rightarrow \infty} E_t(Y_{t+h} | \epsilon_{\theta,t+j} = 0, j = 1, \dots, h) = \lim_{h \rightarrow \infty} \sum_{j=0}^h \theta_{1,t}^j \theta_{0,t} = (I - \theta_{1,t})^{-1} \theta_{0,t}. \quad (5)$$

Since this measure of trend depends only on the current estimates of  $\theta_{0,t}$  and  $\theta_{1,t}$ , it does not require any simulation for future shocks and we can obtain draws from the predictive density fast. However, it takes into account only parameter uncertainty and may underestimate uncertainty about trend due to future shocks to time-varying coefficients. For this reason, this notion of the trend may not be aligned well with near-term and medium-term risk analysis, in which shock uncertainty can play a sizeable role.

### 3.2 Properties of predictive distributions from the TVP-VAR model

All the shocks in the TVP-VAR model follow symmetric normal distributions. Since a normal distribution has a zero skewness and a low kurtosis, it may be argued that the predictive density of the TVP-VAR model inherits these properties of the normal distribution. For example, Cogley, Morozov, and Sargent (2005) mention that the normality assumption would build in a lot of symmetry in forecasts, ignoring the skewness that can arise from asymmetric risks. While this can

be a valid point for one-step ahead forecasts, I will show that the concern is not so critical for trend estimates that are constructed from long-horizon forecasts. The main reason is that the nonlinear dependence of  $Y_{t+h}$  on  $\theta_t$  in the TVP-VAR model generates the departure of a long-horizon predictive density from normality.<sup>10</sup> Under the random-walk hypothesis for  $\theta_t$ , the  $h$ -step ahead forecast error for  $Y_t$  contains the  $h$ -th order polynomial for  $\epsilon_{\theta,t+1}$  because it involves  $\prod_{j=1}^h \theta_{1,t+j}$ . Hence, for  $h \geq 2$ , higher order moments of  $Y_{t+h} - E_t(Y_{t+h})$  must include higher order moments such as  $\epsilon_{\theta,t+1}^h$ . Hence, although  $\epsilon_{\theta,t+1}$  itself is normally distributed,  $Y_{t+h} - E_t(Y_{t+h})$  is not normally distributed for  $h \geq 2$ , generating non-zero skewness.<sup>11</sup> For example, for  $h = 2$ ,  $Y_{t+2} - E_t(Y_{t+2})$  is characterized by the mixture of normal distributions and non-normal distributions such as chi-square distributions. Having a non-normal distribution as a component allows the predictive density to generate a time-varying non-zero skewness.

To illustrate the idea, let's consider the following simple TVP-AR(1) example. A scalar variable  $Y_t$  follows the process below:

$$Y_t = \theta_t Y_{t-1} + u_t, \quad u_t \sim \mathcal{N}(0, \sigma_u^2), \quad (6)$$

$$\theta_t = \theta_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2). \quad (7)$$

The one-step ahead forecast and its' predictive density are given by

$$E_t(Y_{t+1}) = \theta_t Y_t, \quad Y_{t+1} - E_t(Y_{t+1}) = Y_t \epsilon_{t+1} + u_{t+1} \sim \mathcal{N}(0, Y_t^2 \sigma_\epsilon^2 + \sigma_u^2). \quad (8)$$

The one-step ahead forecast error follows a mixture of normal distribution. So it can generate the time-varying volatility and kurtosis but cannot account for a non-zero skewness. However, once we extend the forecast horizon beyond one-step, the predictive density from the TVP-AR(1) model becomes more complicated and can generate a time-varying non-zero skewness. Let's see the two-step ahead forecast

<sup>10</sup>Canova (2007) makes the same observation by contrasting the predictive density of a one-step ahead forecast with that of a two-step ahead forecast from a TVP-VAR model.

<sup>11</sup>This mechanism can be related to the crucial role of the multiplicative stochastic return on capital that generates the non-zero skewness in wealth distribution as mentioned by Benhabib et al. (2011). They show that even if the stochastic return on capital income follows a symmetric distribution, its' cumulative impact on wealth can create the non-zero skewness in wealth distribution.

error for  $Y_t$  as an example.

$$Y_{t+2} - E_t(Y_{t+2}) = Y_t(\epsilon_{t+1}^2 - \sigma_\epsilon^2) + (Y_t\epsilon_{t+2} + u_{t+1})\epsilon_{t+1} + u_{t+1}\epsilon_{t+2} + \theta_t Y_t(\epsilon_{t+2} + 2\epsilon_{t+1}) + u_{t+2} + \theta_t u_{t+1}. \quad (9)$$

Hence, the distribution of the forecast error for  $Y_{t+2}$  conditional on the information set at time  $t$  is the sum of the mixture of normal distributions and non-normal distributions including products of normal random variables. These non-normal distributions allow  $Y_{t+2}$  to have a non-zero skewness. After some algebraic manipulations with  $\sigma_u^2 = 1, \sigma_\epsilon = k$ , one can calculate higher-order moments explicitly.

$$\begin{aligned} E_t(Y_{t+2} - E_t(Y_{t+2}))^2 &= 5Y_t^2\theta_t^2k + 3Y_t^2k^2 + \theta_t^2 + 2k + 1 \\ E_t(Y_{t+2} - E_t(Y_{t+2}))^3 &= 6Y_t^3\theta_t^4k + 9Y_t^3\theta_t^3k^2 + 9Y_t^3\theta_t^2k^2 \\ &\quad + 18Y_t\theta_t^2k + 14Y_t^3k^3 + 12Y_tk^2. \end{aligned} \quad (10)$$

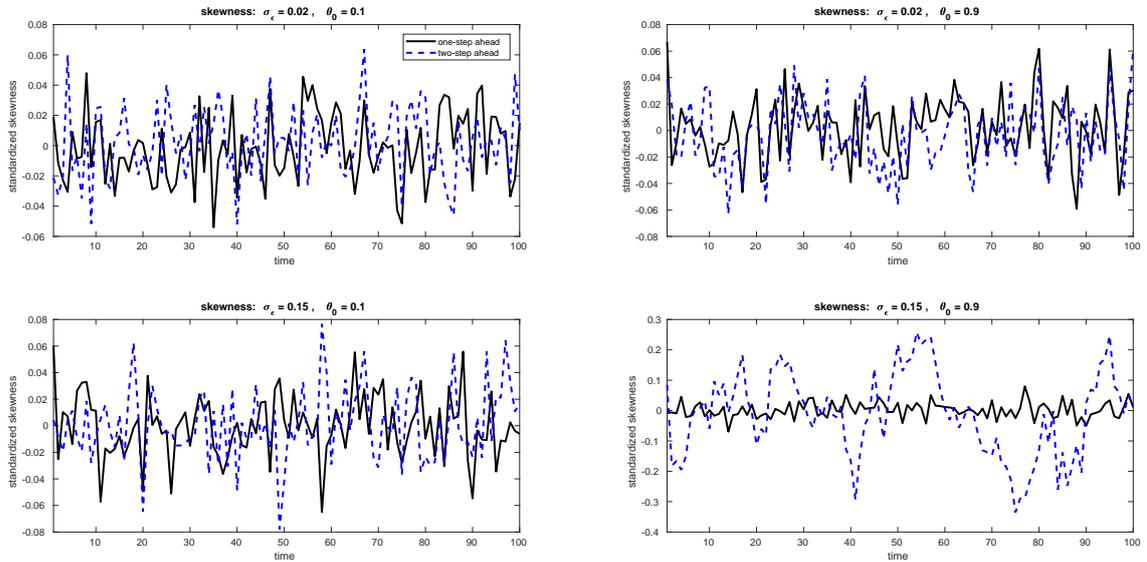
As seen from the above expression, the time-varying skewness of long-horizon forecasts crucially depends on the initial coefficient ( $\theta_0$ ). If  $\theta_0$  is close to zero, even long-horizon forecasts are unlikely to generate a significant skewness because  $\sigma_\epsilon$  is typically much smaller than  $\sigma_u$ . However, if  $\theta_0$  is close to one, long-horizon forecasts can generate a substantial degree of non-zero skewness. Figure 3 shows the time-varying skewness for both one-step ahead and two-step ahead forecasts from the simulated TVP-AR(1) model based on four different combinations of  $\theta_0$  and  $\sigma_\epsilon$ . As the initial value of the AR coefficient ( $\theta_0$ ) implies more persistence in the process and the time-variation in the coefficients ( $\sigma_\epsilon$ ) is bigger, two-step ahead forecasts from the TVP-AR(1) model tend to imply a much greater degree of non-zero skewness than one-step ahead forecasts.<sup>12</sup>

To sum it up, the predictive distributions of long-horizon forecasts in the TVP-VAR model can depart substantially from symmetric and normal distributions

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<sup>12</sup>The nonlinear and non-normal properties of the predictive density for trend are preserved even in the case that I use  $\bar{Y}_t$  as a proxy for trend. Since  $\bar{Y}_t$  is the sum of products between  $\theta_{0,t}$  and  $\theta_{1,t}$ , it is not normally distributed although  $\theta_{0,t}$  and  $\theta_{1,t}$  are normally distributed. In the TVP-AR(1) example,  $\bar{Y}_t$  can be defined by  $\frac{\theta_{0,t}}{1-\theta_{1,t}}$ . Conditional on data ( $Y^t$ ),  $\theta_{0,t}$  and  $1-\theta_{1,t}$  are normally distributed because they are unobserved states in the linear and Gaussian state space model. However,  $\bar{Y}_t$  is the ratio of two correlated normal random variables. Hinkley (1969) provides the exact distribution of this ratio, which is clearly non-normal. In fact, if the numerator and the denominator are independent normal random variables, the ratio follow Cauchy distribution which is heavily fat tailed and does not have finite moments.

Figure 3: TIME-VARYING SKEWNESS: TVP-AR(1) EXAMPLE



*Notes:* Skewness moments are calculated using 10,000 simulated trajectories from the TVP-AR(1) model described in the main text.

unlike those of one-step ahead forecasts. For this reason, the TVP-VAR model can be a suitable framework to study tail risks of macroeconomic variables, which have received much attention since the Great recession and the prolonged period of the zero lower bound in the United States.

### 3.3 Exponential tilting using survey forecasts

One downside of VAR-based forecasts is that the information set included in the VAR may be limited. Expanding the information set by adopting a longer lag structure or more variables can be quite demanding in practice because it can increase the number of parameters to be estimated dramatically. A more efficient way to introduce additional information into VAR-based forecasts is to tilt a predictive density from the model to match information outside the model. For example, Krüger et al. (2017) combine medium-term Bayesian VAR forecasts with external nowcasts using exponential tilting which modifies the baseline predictive density from the VAR to match certain moment conditions from external nowcasts.

They show that exponential tilting improves the forecast accuracy. In addition, Cogley, Morozov, and Sargent (2005) argue that tilting the model-based predictive density to match forecasts from monetary policymakers can generate a more policy-relevant assessment of tail risks.

In this paper, I exponentially tilt the original predictive density of  $\bar{Y}_{t,t+h+1:t+2h}$  to minimize the distance between the predictive moments computed from model estimates of  $[\vartheta, S^T]$  and empirical moments from survey data observations. Survey data do not directly provide observations for  $Y_{t,t+h+1:t+2h}$  but may provide observations for  $Y_{t,t+1}$  and  $Y_{t,t+1:t+h}$ . Since I simulate the predictive density of  $\bar{Y}_{t,t+h+1:t+2h}$  through multiple trajectories of  $Y_{t,t+1:t+2h}$ , I can tilt the predictive density of  $\bar{Y}_{t,t+h+1:t+2h}$  by matching moments from the model-based forecasts of  $Y_{t,t+1}$  and  $Y_{t,t+1:t+h}$  with those from survey forecasts. The practical implementation of the exponential tilting relies on the following method developed by Robertson, Tallman, and Whitemann (2005).

Suppose that the original predictive density is represented by  $N$  equally likely long-term forecasts  $\bar{Y}_{t,t+h+1:t+2h}^i$  for  $i = 1, \dots, N$ . Let's assume that survey data evidence for  $\frac{\sum_{j=1}^h E_{t,i}(Y_{t+j})}{h} = Y_{t,t+h,i}^S$  is available for each forecaster  $i$ . One can find out the new predictive density  $p^*(\bar{Y}_{t,t+h+1:t+2h})$  by minimizing the distance between the two predictive densities based on the Kullback-Leibler information criterion (KLIC) subject to the restriction that cross-sectional means and variances of forecasts from the survey data are matched under the new predictive density. The problem can be formalized as follows:

$$\begin{aligned}
p^*(\bar{Y}_{t,t+h+1:t+2h}) &= \operatorname{argmin} K(p, p^*) = \int \ln \left[ \frac{p^*(\bar{Y}_{t,t+h+1:t+2h}|Y^t)}{p(\bar{Y}_{t,t+h+1:t+2h}|Y^t)} \right] p^*(\bar{Y}_{t,t+h+1:t+2h}|Y^t) dY_{t+h+1}^{t+2h}, \\
&\text{subject to } \int \frac{\sum_{j=1}^h Y_{t+j}}{h} p^*\left(\frac{\sum_{j=1}^h Y_{t+j}}{h} | Y^t\right) dY_{t+j} = E_t(Y_{t,t+h}^S), \\
&\int \left( \frac{\sum_{j=1}^h Y_{t+j}}{h} - E_t(Y_{t,t+h}^S) \right)^2 p^*\left(\frac{\sum_{j=1}^h Y_{t+j}}{h} | Y^t\right) dY_{t+j} = V_t(Y_{t,t+h}^S),
\end{aligned} \tag{11}$$

where  $E_t(Y_{t,t+h}^S)$  and  $V_t(Y_{t,t+h}^S)$  are cross-sectional means and variances of survey participants' forecasts. The solution to this problem is characterized by probability

weights  $p_i^*$  for  $(i = 1, \dots, N)$ , where  $p_i^*$  is defined by<sup>13</sup>

$$p_i^* = \frac{e^{\gamma' Y_{t+h}^i}}{\sum_{i=1}^N e^{\gamma' Y_{t+h}^i}},$$

$$\gamma = \underset{\gamma}{\operatorname{argmin}} \sum_{i=1}^N e^{\gamma_1' \left( \frac{\sum_{j=1}^h Y_{t+j}^i}{h} - E_t(Y_{t,t+h}^S) \right) + \gamma_2' \left( \left( \frac{\sum_{j=1}^h Y_{t+j}^i}{h} - E_t(Y_{t,t+h}^S) \right)^2 - V_t(Y_{t,t+h}) \right)}, \quad (12)$$

$$\gamma = [\gamma_1, \gamma_2].$$

The new predictive density can be represented by resampling the existing forecasts using  $(p_1^*, \dots, p_N^*)$  as new probability weights. The  $h$ -step ahead stagnation risk such as the probability that inflation and consumption growth both fall below 0 can be evaluated using two different densities,  $p(\pi_{t,t+h} < 0, \Delta c_{t,t+h} < 0)$ , and  $p^*(\pi_{t,t+h} < 0, \Delta c_{t,t+h} < 0)$ .

### 3.4 Density evaluation

The use of tilting and different priors for the time-variation in coefficients generate four different predictive distributions for  $Y_{t,t+h}$ . Since the focus in this paper is stagnation risk, I compare density forecasts using quantile-weighted scoring rules suggested by Gneiting and Ranjan (2011). Given the fact that the survey data are available only for the recent few decades, evaluating the accuracy of out-of-sample density forecasts for trend whose forecast horizon spans to 20 years is not so reliable. Therefore, I focus on in-sample performance of density forecasts to discriminate different specifications. The following quantile weighted loss function that penalizes more heavily the inaccuracy in predicting negative tail events is particularly relevant for assessing stagnation risk.

$$QS_\alpha(F^{-1}(\alpha), y) = 2(\mathcal{I}\{y \leq F^{-1}(\alpha)\} - \alpha)(F^{-1}(\alpha) - y), \quad (13)$$

$$S(p, y) = \int_0^1 QS_\alpha(F^{-1}(\alpha), y)(1 - \alpha)^2 d\alpha. \quad (14)$$

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<sup>13</sup>The appendix provides the derivation of this solution.

where  $F$  is the cumulative distribution function for  $y$  based on the predictive density  $p$  and  $F^{-1}(\alpha)$  is the  $\alpha$ -th quantile of  $y$ . The above quantile-weighted loss function is a variation of the continuous ranked probability score (CRPS) function that is often used to measure the overall accuracy of the density forecast.

$$\begin{aligned} CRPS(p, y) &= \int_{-\infty}^{\infty} (F(z) - \mathcal{I}\{y \leq z\})^2 dz, \\ &= 2(\mathcal{I}\{y \leq F^{-1}(\alpha)\} - \alpha)(F^{-1}(\alpha) - y), \end{aligned} \tag{15}$$

where  $\mathcal{I}$  is an indicator function that takes the value of 1 if the condition inside the parenthesis is true.

If we have forecasts from multiple predictive distributions, we can test the equal accuracy of two predictive distributions using the Diebold-Mariano (1995) test statistic using estimates of the loss function from different predictive distributions.

$$\begin{aligned} DM(p_1, p_2) &= \sqrt{n} \frac{\bar{S}(p_1, y) - \bar{S}(p_2, y)}{\bar{\sigma}_n} \sim \mathcal{N}(0, 1) \\ \bar{S}(p, y) &= \frac{\sum_{t=1}^{n-h} S(p(y_{t,t+h}), y_{t+h})}{n-h}, \\ \bar{\sigma}_n &= V(\sqrt{n}(\bar{S}(p_1, y) - \bar{S}(p_2, y))). \end{aligned} \tag{16}$$

The null hypothesis is the equal accuracy of two density forecasts. If  $p_2$  generate more (less) accurate forecasts, the DM test statistic will be positive (negative).

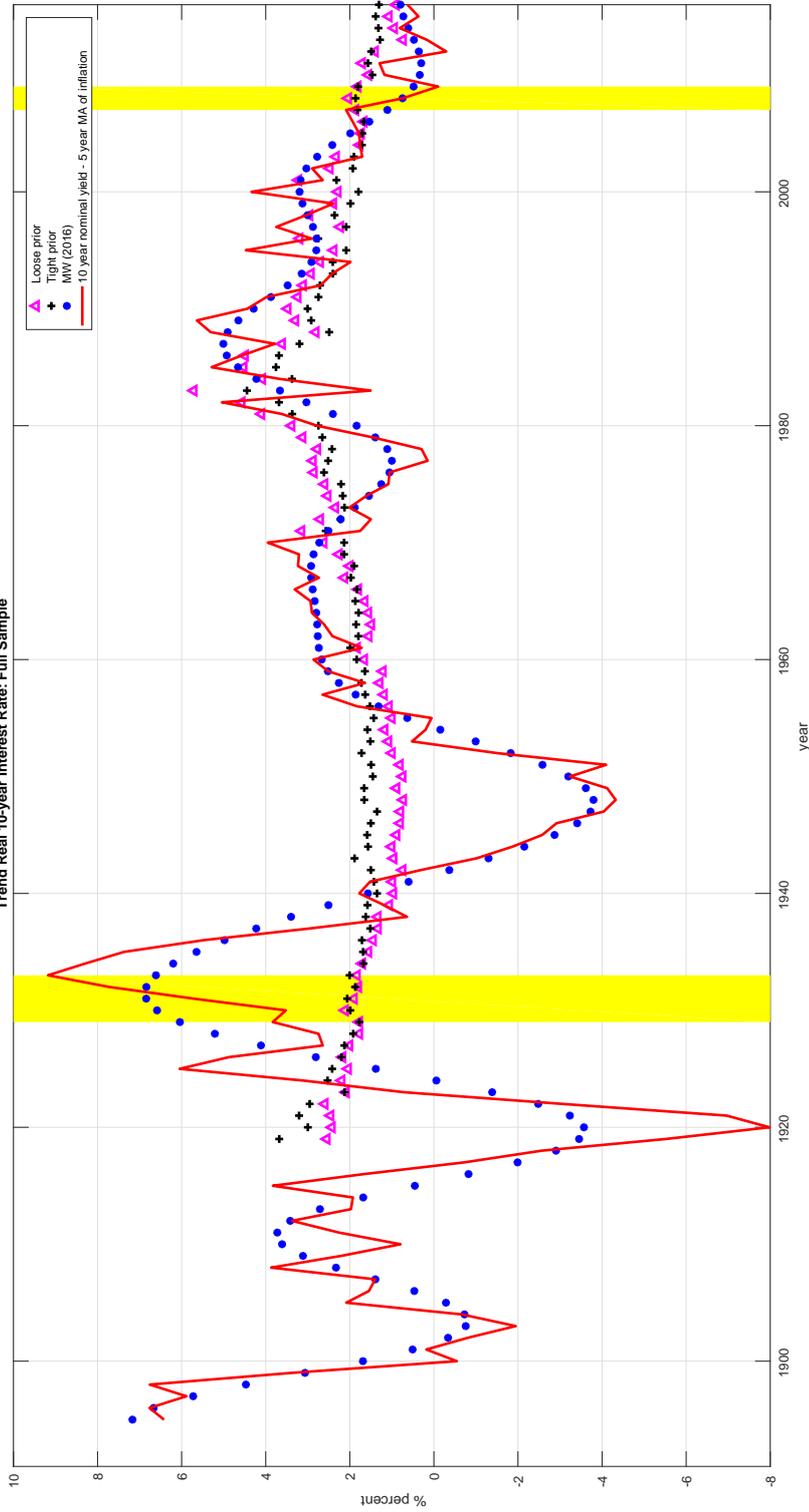
## 4 Empirical Analysis of the U.S. Data

### 4.1 Trend real interest rate

Long-run historical data for the real interest rate are lacking in the U.S. because inflation-indexed government bonds were created only from the late 1990s. However, we can obtain an ex-post measure of the real interest rate using the ten-year nominal interest rate and inflation used in the TVP-VAR model. For example, CEA (2015) constructs the real interest rate defined by the nominal ten-year bond

yield minus the five-year moving average of current and past consumer price inflation going back to the late 19th century. As shown in Figure 4, this ex-post measure exhibits large swings during the period of 1890-2016. The volatility of the real interest rate was much higher during the pre-WWII period. In particular, the real interest rate fluctuated between -8 percent and 9 percent during the interwar period (1919-1939). To obtain the trend component that isolates low-frequency variations of the real interest rate, I use the method suggested in Müller and Watson (2017, MW hereafter). The trend component of the real interest rate obtained by MW (2017) extracts fluctuations with cycles of 16 years and longer.

Figure 4: TREND LONG-TERM REAL INTEREST RATE: FULL SAMPLE



*Notes:* The ex-post measure of the real interest rate is defined by the ten-year nominal bond yield minus the five-year moving average of current and past inflation. The data are from 1891 to 2016. MW (2017) denotes the trend estimate of the ex-post measure of the real interest rate that isolates movements of cycles longer than 16 years based on Müller and Watson (2017). Two median estimates from the TVP-VAR model with different priors are plotted. Shaded areas represent the Great Depression (1929-1933) and the Great Recession (2007-2009) periods.

Table 1: CORRELATION AMONG VARIOUS MEASURES OF THE TREND REAL INTEREST RATE

Variables	Sample Period	Correlation Coefficient
MW (2017) trend, TVP-VAR with tight prior	1919-2016	0.194
MW (2017) trend, TVP-VAR with loose prior	1919-2016	0.409
MW (2017) trend, TVP-VAR with tight prior	1960-2016	0.626
MW (2017) trend, TVP-VAR with loose prior	1960-2016	0.6064

*Notes:* MW (2017) denotes the trend estimate of the ex-post measure of the real interest rate that isolates movements of cycles longer than 16 years based on Müller and Watson (2017). The trend from the TVP-VAR model is the posterior median estimate.

Figure 4 shows the MW (2017) trend with the raw data on the ex-post measure as well as median estimates of two versions of the TVP-VAR model with different priors. Like the ex-post measure, the MW (2017) trend was also more volatile during the pre-WWII period but became less volatile during the post-WWII period. In line with the recent literature, the MW (2017) trend shows the gradual decline since the mid-1980s, which is also observed in the two measures of the trend real interest rate from the TVP-VAR model. However, unlike the MW (2017) trend, the trend real interest rate from the TVP-VAR model was not more volatile during the pre-WWII period, suggesting that the TVP-VAR model absorbs the large fluctuations of the nominal interest rate and inflation during the pre-WWII period largely by the time-varying volatility of temporary shocks.<sup>14</sup> Table 1 reports the correlation between the MW (2017) trend and median estimated of the trend from the TVP-VAR model. While the estimated trend from the model with the tight prior is only weakly correlated with the MW (2017) trend, the one from the model with the loose prior is moderately correlated. For both measures, the correlation increases significantly if I include only the period since 1960.

For a more recent period since the mid-1980s, median estimates from the TVP-VAR model are significantly correlated with empirical measures of the short-term

<sup>14</sup>Although one may argue that the assumption of the constant volatility of innovations to VAR coefficients in the TVP-VAR model forces this result, the robustness of this result to the use of different priors for the volatility of innovations to VAR coefficients suggests that it cannot be entirely attributed to the assumption about the volatility of innovations to VAR coefficients.

Table 2: CORRELATION BETWEEN THE TREND LONG-TERM REAL INTEREST RATE AND THE SHORT-TERM NATURAL REAL INTEREST RATE

Variables	Correlation Coefficient
HLW (2016), DGGT (2017)	0.834
HLW (2016), TVP-VAR with tight prior	0.7419
HLW (2016), TVP-VAR with loose prior	0.8126
HLW (2016), TVP-VAR with tight prior and tilting	0.7433
HLW (2016), TVP-VAR with loose prior and tilting	0.7533
DGGT (2017), TVP-VAR with tight prior	0.5306
DGGT (2017), TVP-VAR with loose prior	0.6355
DGGT (2017), TVP-VAR with tight prior and tilting	0.6707
DGGT (2017), TVP-VAR with loose prior and tilting	0.6828

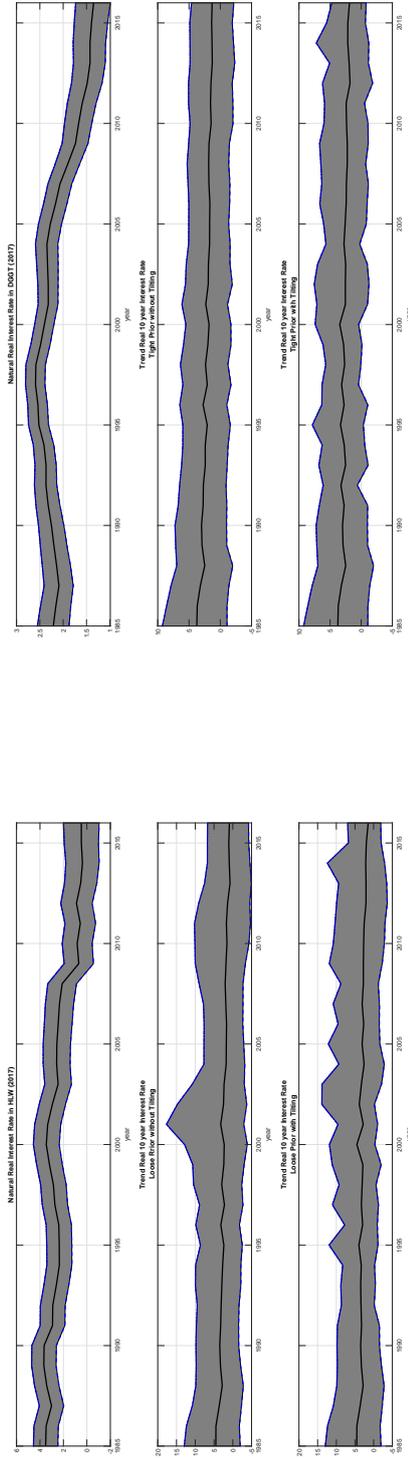
*Notes:* HLW (2017) refers the natural real interest rate estimates from Holston et al. (2017). DGGT (2017) refers the natural real interest rate estimates from Del Negro et al. (2017). The trend from the TVP-VAR model is the posterior median estimate.

natural real interest rate from Holston et al. (2017) and Del Negro et al. (2017) as shown in Table 2. The high correlation between the trend long-term real interest rate measure from the TVP-VAR model and the short-term natural real interest rate indicates that the term premium is unlikely to be a major driver of the trend long-term real interest rate. Since Del Negro et al. (2017) also use the survey data in the estimation, the estimated trend from the TVP-VAR model with tilting is more highly correlated with the measure in Del Negro et al. (2017) than the one without tilting.

In spite of similarities of point estimates, the estimated density for the trend real interest rate from the TVP-VAR model is quite different from Holston et al. (2017) and Del Negro et al. (2017) in terms of higher order moments. As Figure 5 shows, the uncertainty band of the trend real interest rate from the TVP-VAR model is much wider than the one from Del Negro et al. (2017). While tilting affects the location of the median estimate of the trend, it does not reduce the uncertainty band significantly. Average uncertainty bands in Table 3 for various trend estimates illustrate this point. The limited role of tilting in reducing the uncertainty band of the predictive distribution can be explained by the fact that trend estimates consider longer horizon forecasts that are not included in survey

forecasts. In fact, at forecast horizons that correspond to the survey data, I find that tilting tends to reduce the uncertainty range by more than 20%. In addition, the uncertainty range is monotonically increasing at longer horizons as the departure from the normality becomes larger due to the accumulative impact of past shocks to time-varying coefficients. In fact, if we define the trend as the long-run unconditional mean  $\bar{Y}_t$  that ignores the uncertainty due to future shocks, the uncertainty range substantially shrinks to the level comparable to the one reported in Holston et al. (2017) as noted in Table 3.

Figure 5: NATURAL REAL INTEREST RATE AND TREND LONG-TERM REAL INTEREST RATE: RECENT SAMPLE



Notes: HLW (2017) refers the natural real interest rate estimates from Holston et al. (2017). DGGT (2017) refers the natural real interest rate estimates from Del Negro et al. (2017). The black solid line describes the median estimate while the shaded area corresponds to the 68% confidence interval or posterior credible interval.

Table 3: AVERAGE RANGE OF 68% CONFIDENCE (CREDIBLE) INTERVAL: 1985-2016

Variables	Range
HLW (2017)	2.3292
DGGT (2017)	0.5361
TVP-VAR with tight prior	7.477
TVP-VAR with loose prior	12.8507
TVP-VAR with tight prior and tilting	7.4477
TVP-VAR with loose prior and tilting	12.0577
Unconditional Mean in TVP-VAR with tight prior	0.92
Unconditional Mean in TVP-VAR with loose prior	2.96

*Notes:* HLW (2017) refers the natural real interest rate estimates from Holston et al. (2017). DGGT (2017) refers the natural real interest rate estimates from Del Negro et al. (2017). The trend from the TVP-VAR model is the posterior median estimate. The unconditional mean measure represents the trend estimate from  $\bar{Y}_t$  that considers only parameter uncertainty and ignores future shock uncertainty.

A wider uncertainty range of the trend real interest rate from the TVP-VAR model is not a shortcoming by itself. Since the main reason that we are concerned about the decline in the trend real interest rate is the increased risk of stagnation, we would prefer the estimate of the trend real interest rate that can provide a more accurate density forecast of negative tail outcomes for consumption growth and inflation. For example, the TVP-VAR model with the loose prior may generate more accurate density forecasts for negative tail outcomes for consumption growth and inflation than the counterpart with the tight prior although it has a much wider uncertainty range of the estimated trend real interest rate.

## 4.2 Stagnation risk

Recent interests in the trend real interest rate were originated from the concern for secular stagnation in which both the trend growth and interest rate decrease permanently due to structural factors like demographic changes (Eggertsson et al. (2019); Fischer (2016); Gagnon et al. (2016); Rachel and Summers (2019); Summers (2016)). Gagnon et al. (2016) argue that the population aging alone

can explain almost all of the permanent declines in real GDP growth and the equilibrium real interest rate since 1980, using a life-cycle model calibrated to match demographic changes. Summers (2016) argues that such a decline in the equilibrium interest rate is likely to raise downside risks because monetary policy is less suited to addressing negative shocks when the economy is close to the zero lower bound. Fischer (2016) expresses a similar concern. Eggertsson et al. (2019) and Rachel and Summers (2019) recommend expansionary fiscal policies (e.g., debt-financed government spending on investment) to reduce stagnation risk by increasing the equilibrium real interest rate. However, these papers do not provide empirical estimates of stagnation risk because they rely on the calibrated model that ignores the uncertainty related to the simulated outcomes from their models.

This paper tries to provide empirical estimates of stagnation risk that can incorporate multiple sources of uncertainties of future macroeconomic outcomes. I first define stagnation by the probability that deflation and negative consumption growth are realized at the same time. I consider both near-term (one-year ahead) and long-term (five-year ahead) stagnation risk.<sup>15</sup> Table 4 reports the correlation between median estimates of the trend real interest rate and stagnation risks at different horizons. The decline in the trend real interest rate is correlated with the increased risk of stagnation but the precise magnitude varies somewhat by specifications and horizons. Hence, the finding is at least qualitatively consistent with the concern raised by the secular stagnation hypothesis.<sup>16</sup> Tilting predictive distribution to match moments in the survey data has mixed results, strengthening the correlation for the tight prior but weakening it for the loose prior.

To evaluate how serious stagnation risk has been during the recent period, we need quantitative assessments. Figure 6 shows both the near-term and the long-term stagnation risk during the recent period.<sup>17</sup> Both near-term and long-term

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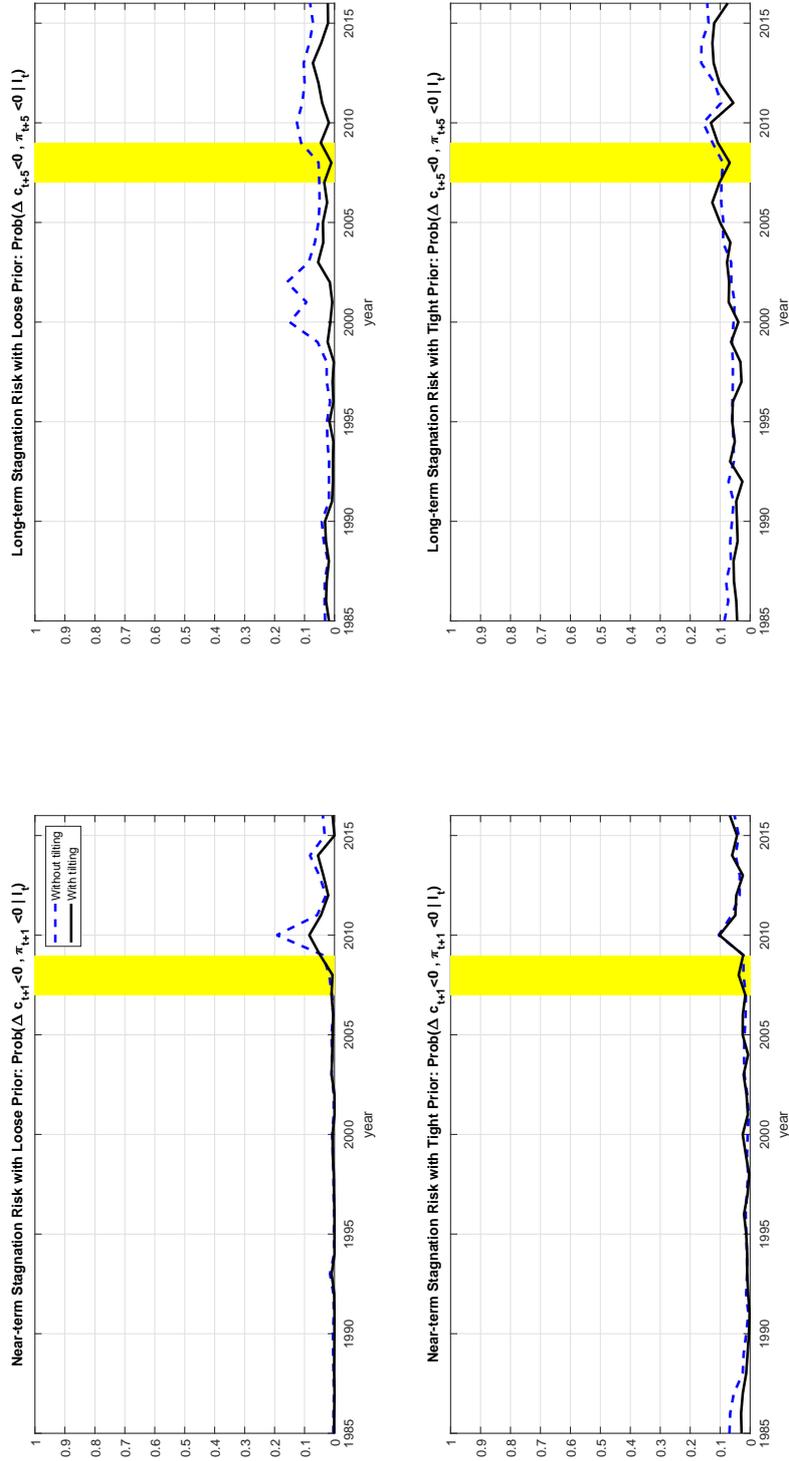
<sup>15</sup>In the appendix, I consider a more moderate version of stagnation that growth and inflation fall below 1 percent.

<sup>16</sup>Using the low-pass filter by Müller and Watson (2017), Lunsford and West (2018) find the long-run correlation between consumption growth and the trend real interest is positive at the 68% confidence interval for the post-WWII period of 1950-2016. But they report that the correlation is much weaker for GDP growth in line with the finding from Hamilton et al. (2016). The correlation with inflation is also somewhat weaker than the one with consumption growth.

<sup>17</sup>For the longer period of 1919-2016, I provide estimates of stagnation risk in the appendix.

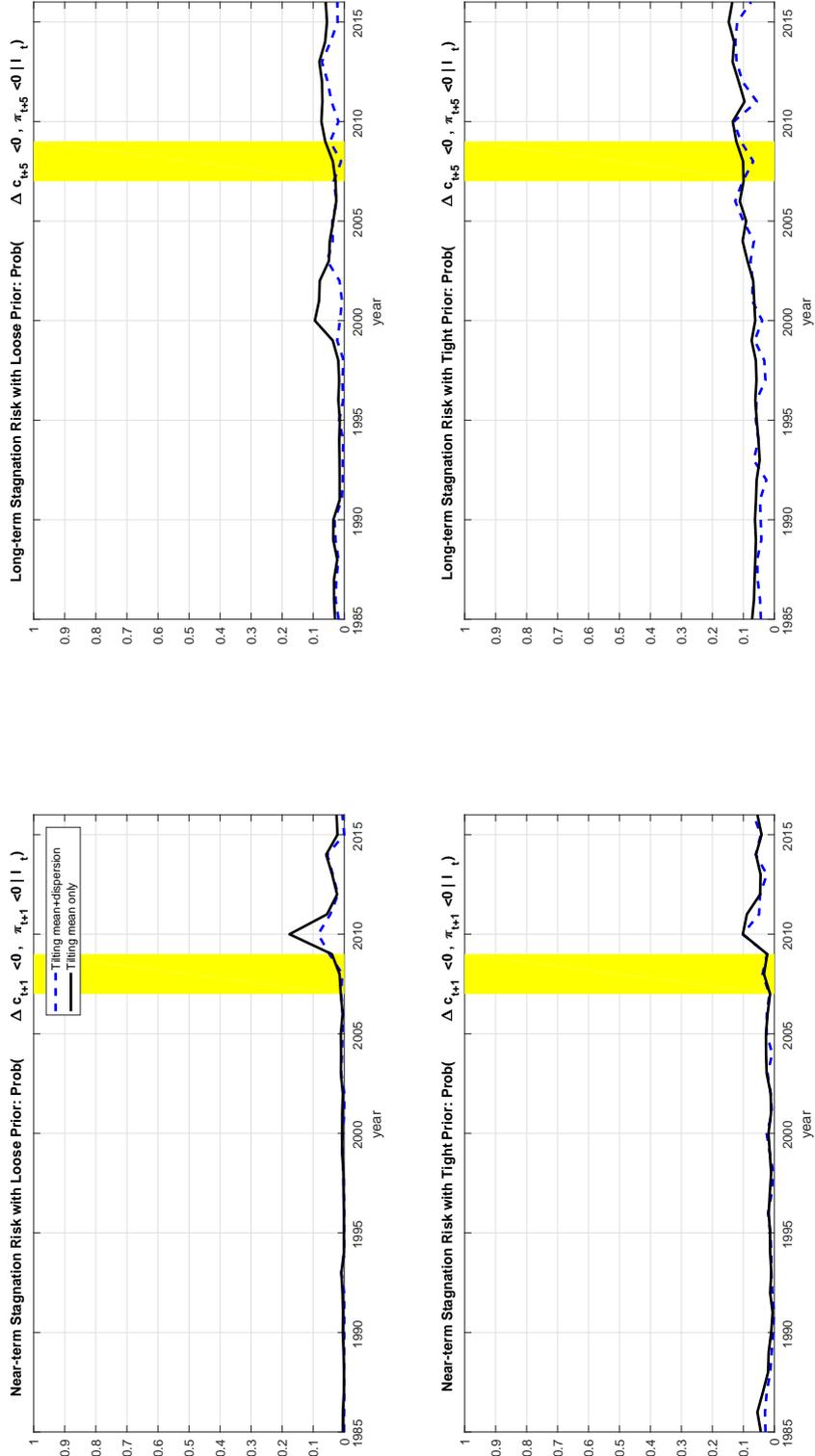
stagnation risk increased during the Great Recession for all the specifications. However, the near-term risk has declined subsequently although the long-term risk has declined more gradually. Tilting tends to reduce stagnation risk but as Figure 7 suggests, that effect is present only if we try to match both cross-sectional means and variances of survey forecasts. If we tilt predictive distributions to match only cross-sectional means of survey forecasts, stagnation risk does not change at all. This finding highlights the relevance of incorporating information from higher order moments of survey forecasts not just the first moment.

Figure 6: STAGNATION RISK: 1985-2016



Notes: The Great Recession (2007-2009) period is shaded.

Figure 7: STAGNATION RISK AND TILTING: 1985-2016



Notes: The Great Recession (2007-2009) period is shaded.

Table 4: CORRELATION BETWEEN THE MEDIAN ESTIMATES OF THE TREND REAL INTEREST RATE AND STAGNATION RISK: 1985-2016

Specification	Near-term Risk	Long-term Risk
TVP-VAR with tight prior	0.0002	-0.5999
TVP-VAR with loose prior	-0.488	-0.5169
TVP-VAR with tight prior and tilting	-0.4113	-0.7062
TVP-VAR with loose prior and tilting	-0.3963	-0.3678

Which estimates of risk are more plausible? To answer this question, I calculate the quantile-weighted probability scores of different predictive distributions. I can score the predictive distribution for one variable at each time. Table 5 reports the ranking of predictive distributions based on the quantile-weighted probability score of the one-year ahead forecasts of consumption growth and inflation. Using the TVP-VAR with the loose prior and no tilting as the baseline specification, I compute the Diebold and Mariano (1995) statistic to test the equal accuracy of two predictive distributions. For consumption growth, no other specifications generate better density forecasts than the baseline specification at a statistically significant level. However, tilting the predictive distribution from the TVP-VAR model with the loose prior generates more accurate density forecasts for inflation than the baseline specification at the 5% significance level. In the case of five-year ahead forecasts shown in Table 6, the quantile-weighted scoring rule implies that the TVP-VAR model with the loose prior and tilting generates more accurate density forecasts than the baseline specification at the 10% significance level for both consumption growth and inflation.<sup>18</sup>

Overall, the quantile-weighted probability scoring rule favors the predictive distributions from the TVP-VAR model with loose prior and tilting both cross-sectional means and variances. According to this specification, the estimated stagnation risk elevated in the aftermath of the Great Recession but subsequently declined significantly. In the beginning of 2016, the near term risk of stagnation was about 0.6% while the long-term risk was 2.2%. Although both numbers were

<sup>18</sup>For inflation, the TVP-VAR model with the tight prior and tilting generates more accurate density forecasts than the baseline specification at the 1% significance level.

Table 5: QUANTILE-WEIGHTED PROBABILITY SCORING OF ONE-YEAR AHEAD FORECASTS: 1985-2016

Specification	Consumption Growth	Inflation
TVP-VAR with tight prior	0.9299 (-8.2238)	0.535 (-0.2162)
TVP-VAR with loose prior	0.5334	0.5888
TVP-VAR with tight prior and tilting (mean only)	0.8879 (-8.6429)	0.5326 (0.2708)
TVP-VAR with loose prior and tilting (mean only)	0.5332 ( 0.2255 )	0.5846 (2.2663)
TVP-VAR with tight prior and tilting (mean+variances)	0.8826 (-8.423)	0.5317 (0.5008)
TVP-VAR with loose prior and tilting (mean+variances)	0.5341 (-0.3869)	0.5837 (1.8381)

*Notes:* Numbers in the parenthesis are the Diebold and Mariano (1995) test statistics on the equality of loss in which the TVP-VAR model with loose prior and no tilting is the baseline specification. A positive number for the Diebold and Mariano (1995) statistic implies that the current specification has lower loss than the baseline specification. The 1%(5%, 10%) critical values for the two sided test is 2.33(1.645, 1.28).

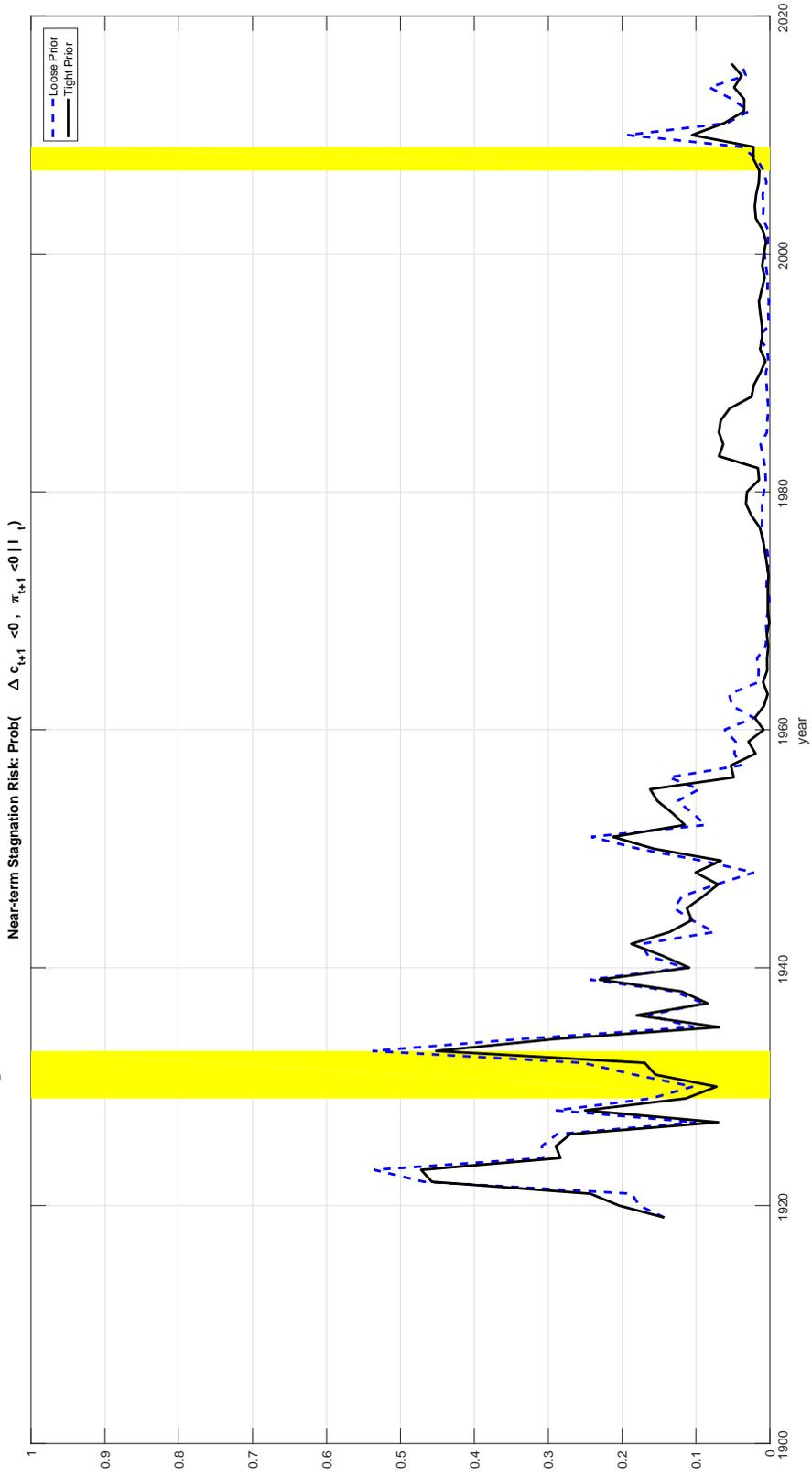
Table 6: QUANTILE-WEIGHTED PROBABILITY SCORING OF FIVE-YEAR AHEAD FORECASTS: 1985-2016

Specification	Consumption Growth	Inflation
TVP-VAR with tight prior	0.7071 (-5.3676)	0.5484 (1.0288)
TVP-VAR with loose prior	0.5023	0.5693
TVP-VAR with tight prior and tilting (mean only)	0.7074 (-5.5092)	0.5505 (1.0438)
TVP-VAR with loose prior and tilting (mean only)	0.5013 (0.3653)	0.5669 (0.8948)
TVP-VAR with tight prior and tilting (mean+variances)	0.7055 (-5.6279)	0.5453 (2.4617)
TVP-VAR with loose prior and tilting (mean+variances)	0.4962 (1.5085)	0.5596 (1.5399)

*Notes:* Numbers in the parenthesis are the Diebold and Mariano (1995) test statistics on the equality of loss in which the TVP-VAR model with loose prior and no tilting is the baseline specification. A positive number for the Diebold and Mariano (1995) statistic implies that the current specification has lower loss than the baseline specification. The 1%(5%, 10%) critical values for the two sided test is 2.33(1.645, 1.28).

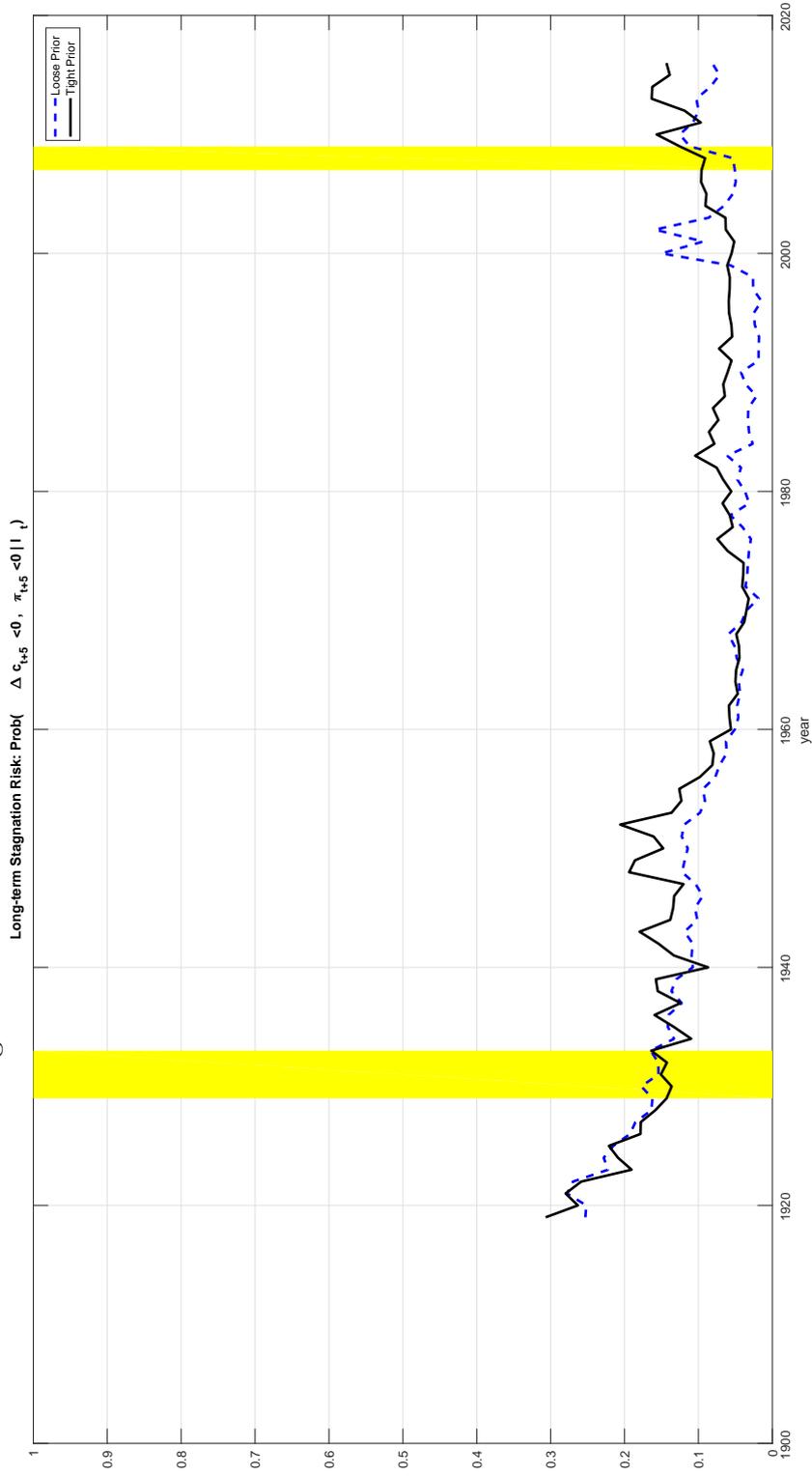
somewhat elevated from the average of 0.3% and 1.9% during the period of 1985-2006, they were much smaller than those from other specifications. For example, stagnation risk was 3.9% in the near-term and 8.1% in the long-term for the TVP-VAR model with loose prior and no tilting in 2016, significantly higher than 0.5% and 4.9%, which are the averages from this specification during the period of 1985-2006. Overall, the estimation results from the TVP-VAR model support the view that stagnation risk increased in the aftermath of the Great Recession but declined subsequently to the relatively low level compared to the pre-WWII period. Estimates of stagnation risk during the longer sample period of 1919-2016 in Figures 8 and 9 suggest that the recent estimates of stagnation risk after the Great Recession are significantly lower than those in 1939 when the secular stagnation hypothesis was originally proposed by Hansen (1939).

Figure 8: NEAR-TERM STAGNATION RISK: 1919-2016



Notes: The Great Depression (1929-1933) and the Great Recession (2007-2009) periods are shaded.

Figure 9: LONG-TERM STAGNATION RISK: 1919-2016



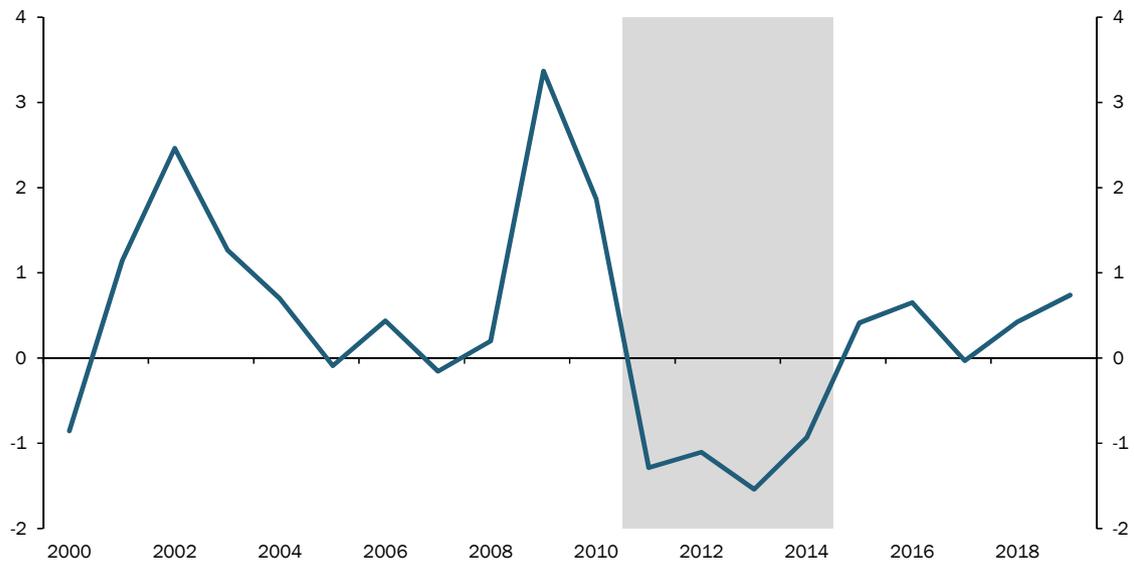
Notes: The Great Depression (1929-1933) and the Great Recession (2007-2009) periods are shaded.

What can explain the decline in stagnation risk that became elevated in the aftermath of the Great Recession? Rachel and Summers (2019) point that major increases in debt as well as social security and healthcare spending offset the impact of a substantial decline in the equilibrium real interest rate on stagnation risk, downplaying the role of unconventional monetary policies such as quantitative easing and forward guidance. In their view, the ZLB constraint is a severe restraint for monetary policy and fiscal policy seems to be a more effective tool to reduce stagnation risk. However, the fiscal impact measure in Sheiner (2014) that takes into account the contribution of trend growth in social spending still indicates that the U.S. fiscal policy was in a contractionary territory between 2011 and 2014 as shown in Figure 10. The stimulus impact from the American Recovery and Reinvestment Act of 2009 was mostly phased out by 2011. With the passage of the Budget Control Act in 2011, the U.S. fiscal policy became less expansionary while unconventional monetary policies such as the Federal Reserve’s large scale asset purchases continued to expand. In addition, Albonico et al. (2017) find that the fiscal policy during the recovery phase from the Great Recession was contractionary in spite of the potentially large multiplier effect of government spending due to the ZLB constraint. They interpret the lack of fiscal impetus during the recovery phase as a missed opportunity.<sup>19</sup>

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<sup>19</sup>The December 2013 post-meeting statement issued by the Federal Open Market Committee (2013) explicitly acknowledged “the extent of federal fiscal retrenchment since the inception of its current asset purchase program” to signal that monetary policy remained accommodative in part to compensate for the lack of fiscal stimulus during this period.

Figure 10: FISCAL IMPACT MEASURE IN THE U.S.: 2000-2019



*Notes:* The first-quarter value of each year is plotted. The source data are from the Hutchins Center on Fiscal and Monetary Policy at the Brookings Institution. The methodology behind this measure is explained in Sheiner (2014). The shaded area represents the period of 2011-2014 when the fiscal policy stance was contractionary, meaning that the government was subtracting from real GDP growth.

Debortoli et al. (2018) offer an alternative view that the ZLB constraint became empirically irrelevant because of aggressive monetary policy responses during and after the Great Recession. Also, Aruoba et al. (2018) suggest that the U.S. can avert the transition to the deflationary steady state in the aftermath of the Great Recession unlike Japan who got trapped in it during the late 1990s. They attribute the difference between the U.S. and Japan to more aggressive policy responses by the Federal Reserve that might have helped in coordinating inflation expectations at its positive target level. The time-variation in stagnation risk from the TVP-VAR model seems to be more consistent with this view because stagnation risk declined more substantially when the predictive density of the model-implied forecasts is tilted to match cross-sectional means and variances of survey forecasts.

## 5 Conclusion

Empirical studies showing a decline in the trend real interest rate during the recent few decades raised a concern that the U.S. economy might have entered into a low growth and low inflation stagnation regime as monetary policy is more frequently constrained by the ZLB constraint on the nominal interest rate due to the lower level of the trend real interest rate. This risk of stagnation has received a lot of attention since growth and inflation in the aftermath of the Great Recession of 2007-2009 were persistently lower despite the historically low interest rate than in the previous recovery. Rigorous empirical analysis of the trend real interest rate with stagnation risk is challenging because both concepts are subject to great uncertainties given a limited sample of the time series data. This paper uses predictive distributions of macro variables from the TVP-VAR model to estimate both the trend real interest rate and stagnation risk. The method can take into account multiple sources of uncertainties as well as the departure from symmetric normal distributions in estimating the trend real interest rate. In addition, the analysis incorporates information from survey forecasts of inflation and the nominal interest rate by tilting predictive distributions of these variables to match cross-sectional means and variances of survey forecasts. I use a quantile-weighted probability scoring rule to rank different predictive distributions from various specification of the TVP-VAR model. Such a rule provides the criterion for the statistical test of the equal accuracy of density forecasts for negative tail events.

According to the estimated TVP-VAR model using long-run historical data in the U.S. going back to the interwar period of 1919-1939, the trend real interest rate has declined over the last few decades and the decline is correlated with the increased risk of stagnation. The correlation is higher for long term (five-year ahead) predictions than near-term (one-year ahead) predictions though the magnitude varies by the precise specification of the TVP-VAR model. Time series estimates of stagnation risk characterized by negative consumption growth and deflation indeed increased during the Great Recession. Although the peak level of stagnation risk in the aftermath of the Great Recession was significantly elevated relative to the post-WWII average, it was much lower than the peak level reached

during the interwar period. And the near-term risk declined quickly one year after the Great Recession ended while the long-term risk declined more sluggishly. The comparison of density forecasts from various specifications of the TVP-VAR model based on the quantile-weighted scoring rule favors the predictive density from the TVP-VAR that has a loose prior for the magnitude of the time variation of VAR coefficients and is tilted to match cross-sectional means and variances from survey forecasts of inflation and the nominal interest rate. The favored specification generates a lower risk of stagnation. For example, the long-term risk of stagnation approaching the pre-crisis (1985-2006) average by 2014 under this specification. The existing literature on the secular stagnation hypothesis emphasizes the role of expansionary fiscal policies in reducing stagnation risk when monetary policy is constrained by the ZLB. However, the decline in the estimated stagnation risk coincides with the ongoing monetary stimulus but contractionary fiscal policies between 2011 and 2014. This finding is consistent with the view that the Federal Reserve’s unconventional policies were effective in mitigating negative tail risks by influencing the private sector’s expectations for inflation and the nominal interest rate.

## Appendix

The appendix first describes the details behind Bayesian estimation of the TVP-VAR model and exponential tilting of the predictive density. Then it provides estimates of the risk of a more moderate version of stagnation where consumption growth and inflation both fall below 1%.

### A.1 The elicitation of prior distributions

As explained in the main text, data from 1891 to 1914 are used to construct prior distributions for parameters and initial values of latent variables. First, I estimate the first-order VAR with constant coefficients and volatilities for the pre-sample data.

$$Y_t = \theta_0 + \theta_1' Y_{t-1} + u_t. \tag{17}$$

The prior mean of  $\Sigma_\theta$  is obtained by shrinking the estimated covariance matrix for the VAR coefficients so that the implied change in trend estimates (measured by long-run means) during the pre-sample period is less than or equal to the standard deviation of each variable in  $Y_t$  for the same period. More specifically, we choose the prior mean of  $\Sigma_\theta$  in the tight prior case to make the following condition hold.

$$T_0 \times E(\bar{Y}_{t+1} - \bar{Y}_t | \Sigma_\theta) = \min[\text{Std}(Y_t)]. \quad (18)$$

The prior mean for  $\Sigma_\theta$  in the loose prior case multiplies the prior mean of  $\Sigma_\theta$  obtained in the above way by 4.

## A.2 The posterior simulation

While deriving the joint posterior distribution of  $(\vartheta, S^T)$  analytically is intractable, posterior distribution of a subset of  $(\vartheta, S^T)$  conditional on all the rest is tractable. The exception is the stochastic volatility in VAR residual shocks for which we do not have the analytical form for the conditional posterior distribution. For that part, I use Metropolis-Hastings algorithm that simulates posterior draws based on the conditional posterior kernel without knowing the exact posterior density. For other parts, I use Gibbs Sampling by taking advantage of the analytically known conditional posterior distribution. Below are some more details of the posterior simulation.

**Step 1: Initialization** Estimate VAR residuals  $(u_t)$  using the sample data by setting  $\theta_{0,t}$  to the sample average of  $Y_t - Y_{t-1}$  and  $\theta_{1,t}$  to the identity matrix. This implicitly assumes a univariate random walk process for each variable in  $Y_t$ . Using squared VAR residuals as proxies for stochastic volatilities, one can draw  $\sigma_{\xi,i}$  from the known conditional posterior distribution which is an inverse gamma distribution.

**Step 2: Metropolis-Hastings algorithm to draw  $D_{ii}^T$**  Using the previous draw for  $B$  and VAR residuals  $u_t$ , one can construct structural shock estimates  $\epsilon_t = Bu_t$ . The variance of  $\epsilon_t$  is  $D_{ii,t}$ . While the exact posterior distribution of  $D_{ii,t}$  conditional on  $(Y^T, \vartheta, \theta^T)$  is not known analytically, the conditional posterior

kernel can be calculated easily as follows:

$$p(D_{ii,t}|\vartheta, Y^T, \theta^T, D_{ii,t-1}, D_{ii,t+1}) \propto p(\varepsilon_{i,t}|D_{ii,t}, \vartheta)p(D_{ii,t}|D_{ii,t-1}, \vartheta)p(D_{ii,t+1}|D_{ii,t}, \vartheta). \quad (19)$$

Under assumptions made in the TVP-VAR model, all the components in the posterior kernel are analytically known and easy to compute.

**Step 3: Draw  $B$**  Given  $D^T$  and  $u^T$ , off-diagonal terms in  $B$  can be computed as the regression coefficients of  $u_t$  on  $\varepsilon_t$ . For example,  $B$  is a  $2 \times 2$  matrix where  $B_{21}$  is the only off-diagonal term. Because  $Bu_t = \varepsilon_t$  and  $B = \begin{pmatrix} 1 & 0 \\ B_{21} & 1 \end{pmatrix}$ , we can run the following regressions.

$$\begin{aligned} u_{1,t} &= \varepsilon_{1,t}, \\ (D_{22,t}^{-0.5}u_{2,t}) &= B_{21}(-D_{22,t}^{-0.5}u_{1,t}) + (D_{22,t}^{-0.5}\varepsilon_{2,t}). \end{aligned} \quad (20)$$

As explained by Cogley and Sargent (2005), the above regressions imply the normal posterior for  $B_{21}$ . Generalizing this, one can draw  $B$  from the exact conditional posterior distribution which is a multivariate normal distribution.

**Step 4: Draw  $\theta^T$**  Given  $\vartheta$ ,  $D^T$ , and  $Y^T$ , the TVP-VAR model is represented by a linear and Gaussian state space model for  $\theta^T$  where  $\theta_t$  is a latent state vector. Using the Kalman filter, one can derive the conditional posterior distribution of  $\theta^T$  that is a normal distribution. Carter and Kohn (1994) provide an algorithm to draw  $\theta^T$  from the conditional posterior distribution by applying the Kalman filter both forwards and backwards recursively.

**Step 5: Draw  $\Sigma_\theta$**  Given  $\theta^T$ , innovations in the evolution of VAR coefficients are known and follow normal distributions. Assuming the inverse-Wishart prior for  $\Sigma_\theta$ , the conditional posterior distribution of  $\Sigma_\theta$  is also inverse-Wishart and easy to simulate. Once  $\Sigma_\theta$  is drawn, go back to **Step 1** and repeatedly draw another posterior draw for  $\vartheta$  and  $S^T$ .

### A.3 Implementing exponential tilting

The goal of exponential tilting is to find out a new predictive density  $\hat{p}$  that

minimizes the KLIC subject to matching moments from the survey data.

$$\hat{p} = \operatorname{argmin} KLIC(\hat{p}, p) = \int \ln\left(\frac{\hat{p}}{p}\right) \hat{p} dY_{t+h+1}^{t+2h}, \quad (21)$$

subject to  $E_t^{\hat{p}}(\bar{Y}_{t+1:t+h}) = E_t(Y_{t,t+h}^s)$ ,  $V_t^{\hat{p}}(\bar{Y}_{t+1:t+h}) = V_t(Y_{t,t+h}^s)$ .

Notice that  $p = \frac{1}{N}$  and  $\hat{p}$  is a probability density such that  $\int \hat{p} dY_{t+h+1}^{Y+2h} = 1$ . Now stack all the three constraints into  $g(Y_{t+1:t+h})$  as follows:

$$g(Y_{t+1:t+h}) = \sum_{i=1}^N \pi_i^* [Y_{t+1:t+h}^i - E_t(Y_{t,t+h}^s), (Y_{t+1:t+h}^i - E_t(Y_{t,t+h}^s))^2 - V_t(Y_{t,t+h}^s)]'. \quad (22)$$

The above minimization problem is solved by two steps: 1) maximize the Lagrangian with respect to Lagrangian multipliers ( $\gamma$ ) and then 2) minimize the Lagrangian with respect to  $\hat{p}$  given  $\gamma$ . To understand these steps, consider the following Lagrangian function.

$$\mathcal{L} = \sum_{i=1}^N N(\ln(\pi_i^*)) \pi_i^* - \gamma' \sum_{i=1}^N \pi_i^* g(\bar{Y}_{t+1:t+h}^i). \quad (23)$$

Given  $\gamma$ , the first-order condition for  $\pi_i^*$  is given by

$$\ln \pi_i^* = -N + \gamma' g(Y_{t+1:t+h}^i). \quad (24)$$

Now, using the fact that  $\sum_{i=1}^N \pi_i^* = 1$ ,  $\pi_i^* = \frac{e^{\gamma' g(Y_{t+1:t+h}^i)}}{\sum_{i=1}^N e^{\gamma' g(Y_{t+1:t+h}^i)}}$ , plug this expression for  $\pi_i^*$  back into  $\mathcal{L}$ . As a result, we can express  $\mathcal{L}$  as the function of only  $\gamma$  noting that all the constraints are binding ( $\sum_{i=1}^N \pi_i^* g(Y_{t+1:t+h}^i) = 0$ ).

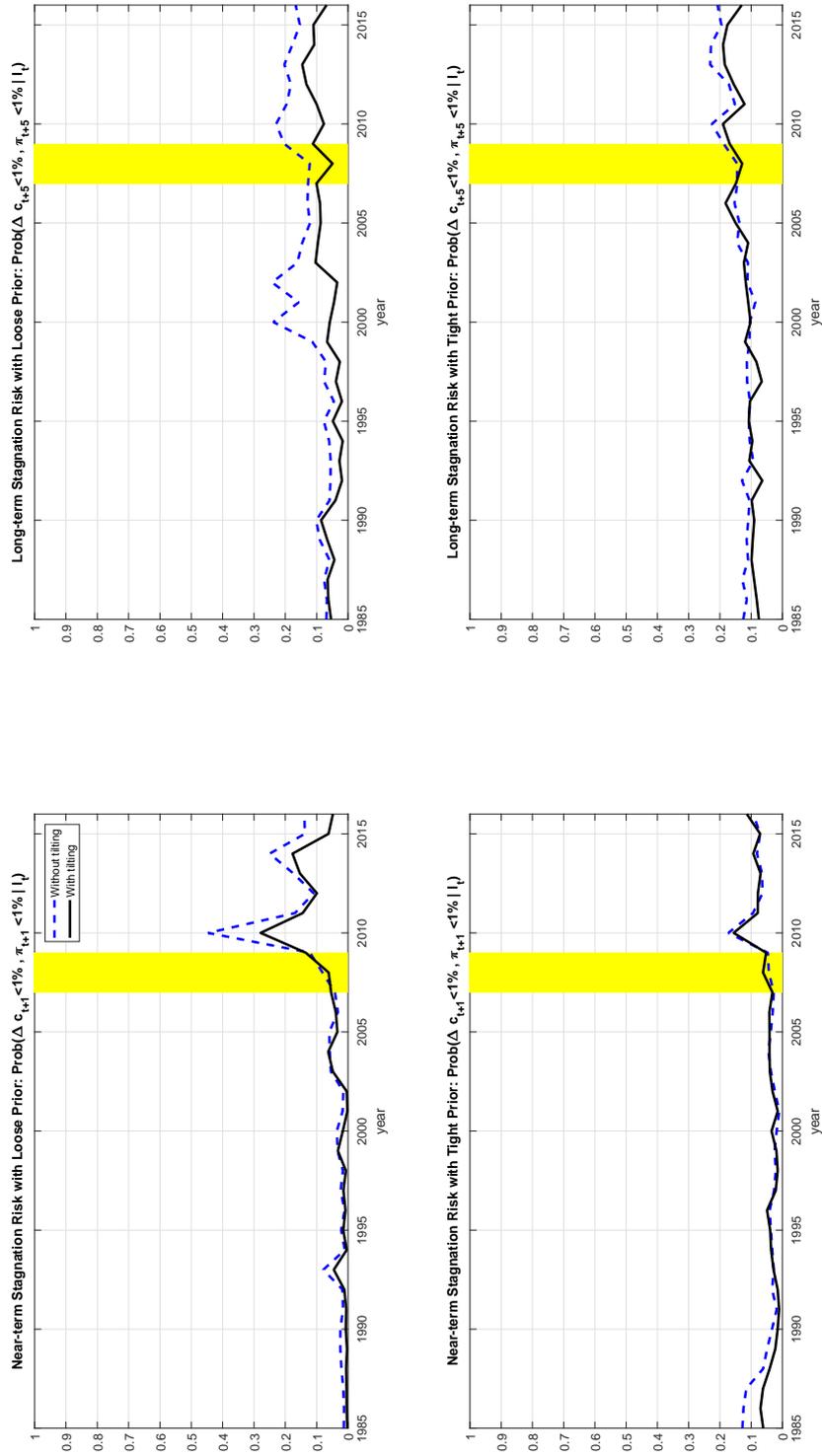
$$\gamma^* = \operatorname{argmax} -N \sum_{i=1}^N \ln\left(\sum_{i=1}^N e^{\gamma' g(Y_{t+1:t+h}^i)}\right) = \operatorname{argmin} \sum_{i=1}^N e^{\gamma' g(Y_{t+1:t+h}^i)}. \quad (25)$$

To implement exponential tilting, first find out  $\gamma^*$  by applying a numerical optimization routine and then calculate  $\pi_i^* = \frac{e^{\gamma^{*'} g(Y_{t+1:t+h}^i)}}{\sum_{i=1}^N e^{\gamma^{*'} g(Y_{t+1:t+h}^i)}}$ .

#### A.4 Risk of a more moderate version of stagnation

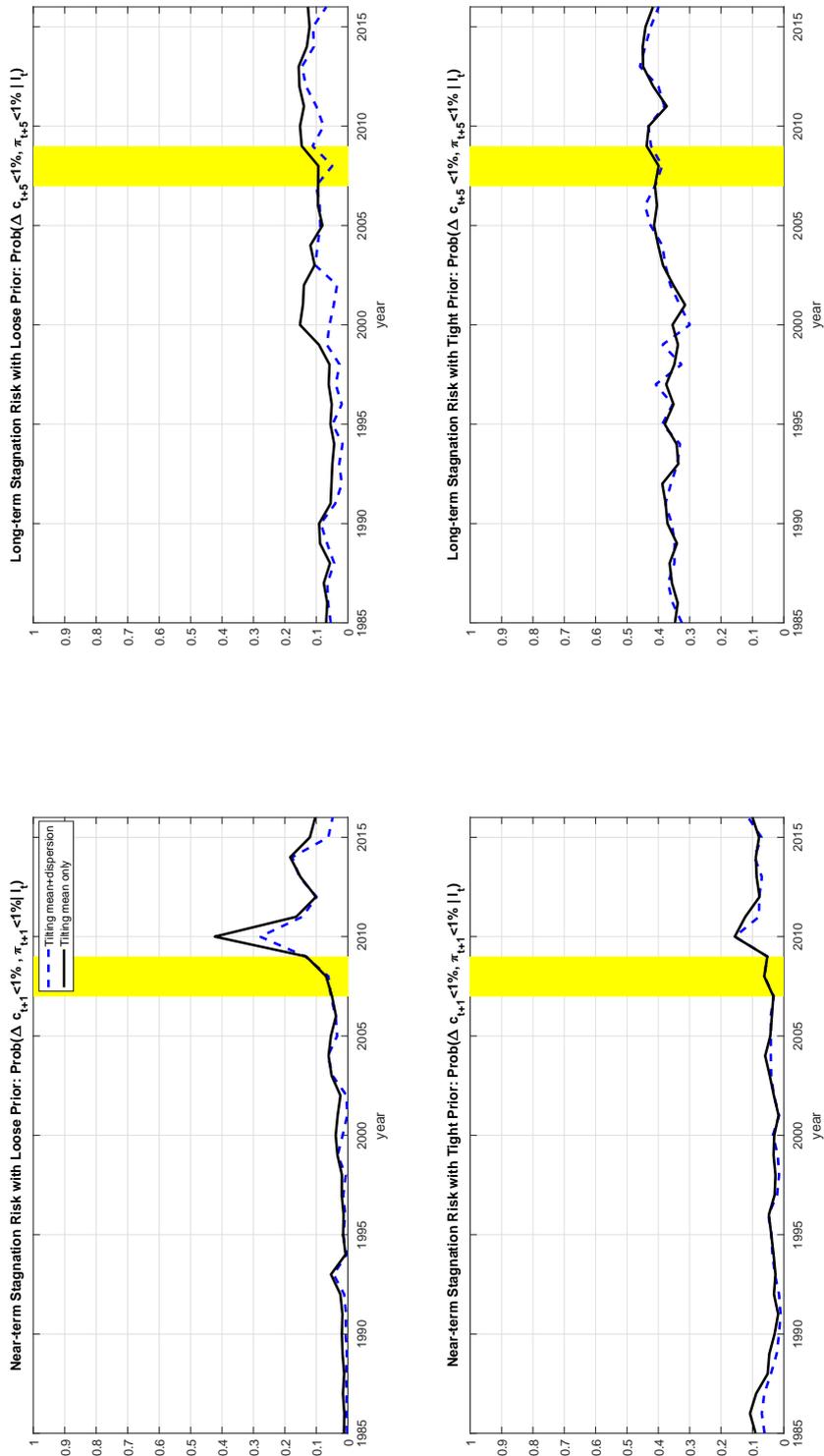
The stagnation scenario considered in the main text is an extreme tail event where both consumption growth and inflation become negative. During the sample period of 1919-2016, such an event happened only in 4 years (1931-1933, 1939) in the pre-WWII period. In this section, I consider a more moderate version of stagnation where consumption growth inflation both fall below 1%. On top of the four years in the pre-WWII period, this event happened in 1962 and 2009. I plot the time-variation of stagnation risk for the moderate version of stagnation in Figures 11-14. They are qualitatively very similar to Figures 6-9 with a more extreme definition of stagnation. Therefore, the main conclusion in this paper appears to be robust to a change in the severity of stagnation.

Figure 11: STAGNATION RISK: 1985-2016



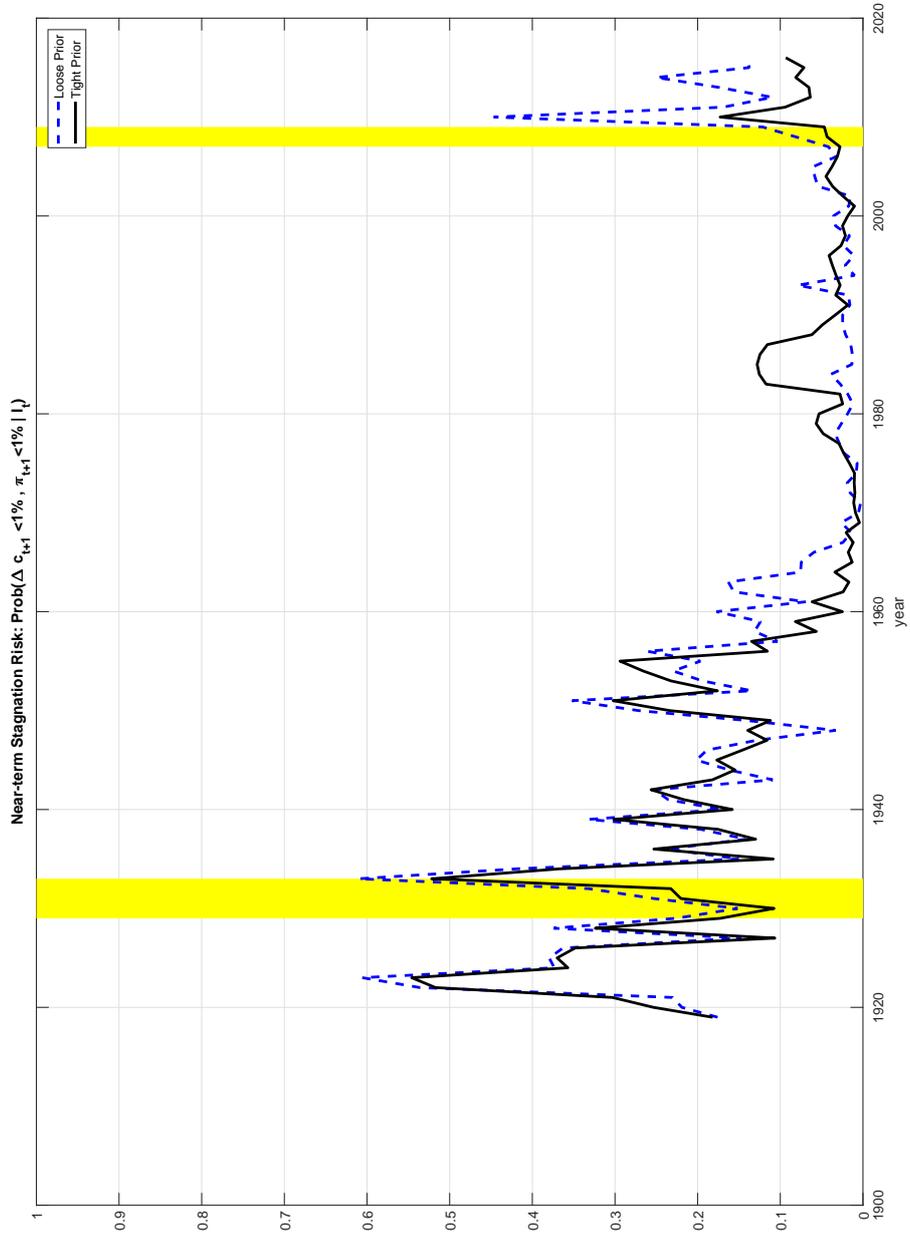
Notes: The Great Recession (2007-2009) period is shaded.

Figure 12: STAGNATION RISK AND TILTING: 1985-2016



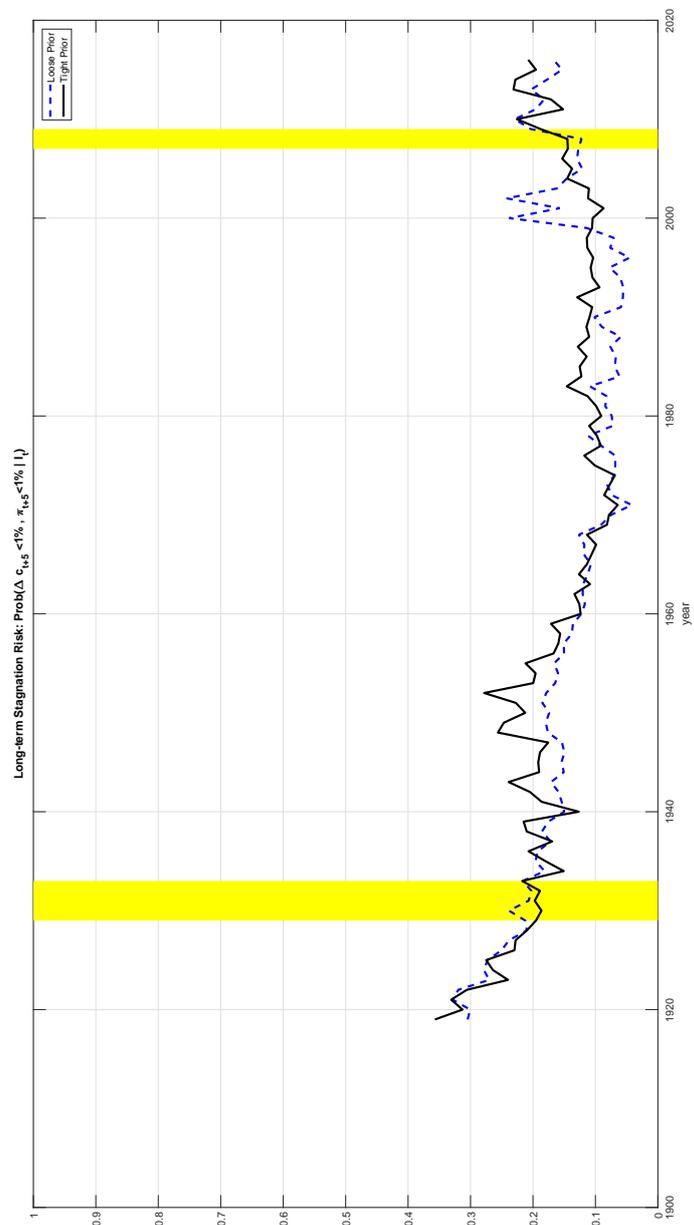
Notes: The Great Recession (2007-2009) period is shaded.

Figure 13: NEAR-TERM STAGNATION RISK: 1919-2016



Notes: The Great Depression (1929-1933) and the Great Recession (2007-2009) periods are shaded.

Figure 14: LONG-TERM STAGNATION RISK: 1919-2016



Notes: The Great Depression (1929-1933) and the Great Recession (2007-2009) periods are shaded.

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