

Forward Guidance, Monetary Policy Uncertainty, and the Term Premium

Technical Appendix*

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*The views expressed herein are solely those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of Kansas City or the Federal Reserve System.

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A Appendix

This appendix provides the derivation of the equations used in Section 2 to motivate the empirical work done in the subsequent sections. We use a second-order approximation to capture the effects of uncertainty.

A.1 Definitions and Household Decisions

The price p of a n -period bond at time t is set using the stochastic discount factor m :

$$p_t^{(n)} = \mathbb{E}_t \left\{ m_{t+1} p_{t+1}^{(n-1)} \right\}$$

Let these bonds be sold at a discount so that gross yield Y is defined as follows:

$$1 = p_t^{(n)} \left(Y_t^{(n)} \right)^n$$

This gives the following relationship for the net yield $y = \log(Y)$:

$$ny_t^{(n)} = -\log \left(p_t^{(n)} \right)$$

Because bonds are sold at a discount, the bond price at maturity is defined as 1:

$$p_{t+k}^{(1)} = \mathbb{E}_t m_{t+k+1}, \forall k$$

Now, assume a representative household which maximizes lifetime expected utility over consumption C_t . The household receives endowment income e_t and can purchase nominal bonds $b_{t+1}^{(n)}$ with maturity n of one to N periods. Denote the aggregate price level P_t .

Formally, the representative household chooses C_{t+s} , and $b_{t+s+1}^{(n)}$ for all bond maturities $n = 1, \dots, N$ and all future periods $s = 0, 1, 2, \dots$ by solving the following problem:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \log(C_{t+s})$$

subject to the intertemporal household budget constraint each period,

$$C_t + \sum_{n=1}^N p_t^{(n)} \frac{b_{t+1}^{(n)}}{P_t} + \leq e_t + \sum_{n=1}^N p_t^{(n-1)} \frac{b_t^{(n)}}{P_t} \quad (\text{with multiplier } \lambda_t).$$

This leads to the following first-order conditions:

$$\frac{1}{C_t} = \lambda_t$$

$$p_t^{(1)} = \mathbb{E}_t \left\{ \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \right\}$$

$$p_t^{(n)} = \mathbb{E}_t \left\{ \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} p_{t+1}^{(n-1)} \right\}$$

Assuming prices are fixed $P_t = P$ simplifies these conditions and allows for tractability.

Following from the bond pricing equation, the stochastic discount factor can now be defined as follows:

$$\mathbb{E}_t m_{t+1} = \mathbb{E}_t \left\{ \beta \frac{\lambda_{t+1}}{\lambda_t} \right\}$$

The one-period gross interest rate R_t is equal to the inverse of one-period bond price. This implies:

$$\frac{1}{R_t} = p_t^{(1)} = \mathbb{E}_t m_{t+1}$$

The risk-neutral bond price q is not affected by the covariance between the discount factor and the future expected bond price, which is also risk-neutral:

$$q_t^{(n)} = \mathbb{E}_t m_{t+1} \mathbb{E}_t q_{t+1}^{(n-1)}$$

Using the equations above, we can rewrite this:

$$q_t^{(n)} = \frac{1}{R_t} \mathbb{E}_t q_{t+1}^{(n-1)}$$

And note how a one-period risk-neutral bond is priced using this relation:

$$q_{t+k}^{(1)} = \frac{1}{R_{t+k}}, \forall k$$

To get second-order approximations of these series, use the following Taylor series centered at zero:

$$\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

$$\log(1+x) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j}$$

Also, note the definitions of variance and covariance:

$$\text{VAR}_t X_{t+1} = \mathbb{E}_t X_{t+1}^2 - (\mathbb{E}_t X_{t+1})^2$$

$$\text{COV}_t X_{t+1} W_{t+1} = \mathbb{E}_t X_{t+1} W_{t+1} - \mathbb{E}_t X_{t+1} \mathbb{E}_t W_{t+1}$$

A.2 Consumption and Yield

We start by using the discount factor implied by the Euler equations to relate bond yields to consumption. This shows that consumption increases as yields decreases and that consumption decreases as future consumption becomes more uncertain.

First, use the Euler equations to price the following bonds:

$$p_t^{(n)} = \mathbb{E}_t \left\{ \beta \frac{C_t}{C_{t+1}} p_{t+1}^{(n-1)} \right\}$$

$$p_{t+1}^{(n-1)} = \mathbb{E}_{t+1} \left\{ \beta \frac{C_{t+1}}{C_{t+2}} p_{t+2}^{(n-2)} \right\}$$

Using substitution, the price of the n -period bond can be further simplified:

$$p_t^{(n)} = \mathbb{E}_t \left\{ \beta \frac{C_t}{C_{t+1}} \mathbb{E}_{t+1} \left\{ \beta \frac{C_{t+1}}{C_{t+2}} p_{t+2}^{(n-2)} \right\} \right\}$$

$$p_t^{(n)} = \mathbb{E}_t \left\{ \mathbb{E}_{t+1} \left\{ \beta^2 \frac{C_t}{C_{t+1}} \frac{C_{t+1}}{C_{t+2}} p_{t+2}^{(n-2)} \right\} \right\}$$

$$p_t^{(n)} = \mathbb{E}_t \left\{ \beta^n \frac{C_t}{C_{t+n}} \right\}$$

Now, move all of the time t variables to the left side of the equation:

$$\frac{p_t^{(n)}}{\beta^n C_t} = \mathbb{E}_t \left\{ \frac{1}{C_{t+n}} \right\}$$

Define $c_t = \log(C_t)$, and set up the equation for the Taylor series approximations:

$$\frac{p_t^{(n)}}{\beta^n C_t} = \mathbb{E}_t \{ \exp(-\log(C_{t+n})) \}$$

$$\frac{p_t^{(n)}}{\beta^n C_t} = \mathbb{E}_t \{ \exp(-c_{t+n}) \}$$

$$\log(p_t^{(n)}) - c_t - n \log(\beta) = \log(\mathbb{E}_t \{ \exp(-c_{t+n}) \})$$

Take two Taylor series approximations. First, for $\exp(\cdot)$; second for $\log(\cdot)$:

$$\log(p_t^{(n)}) - c_t - n \log(\beta) = \log \left(1 + \mathbb{E}_t \left\{ -c_{t+n} + \frac{1}{2} c_{t+n}^2 \right\} \right)$$

$$\log(p_t^{(n)}) - c_t - n \log(\beta) = -\mathbb{E}_t c_{t+n} + \frac{1}{2} \mathbb{E}_t c_{t+n}^2 - \frac{1}{2} (\mathbb{E}_t c_{t+n})^2$$

Use the definition of variance to simplify:

$$\log(p_t^{(n)}) - c_t - n \log(\beta) = -\mathbb{E}_t c_{t+n} + \frac{1}{2} \text{VAR}_t c_{t+n}$$

Finally, use the definition of yield given in the previous section, and solve for c_t :

$$-c_t = -\log(p_t^{(n)}) - \mathbb{E}_t c_{t+n} + \frac{1}{2} \text{VAR}_t c_{t+n} + n \log(\beta)$$

$$-c_t = n y_t^{(n)} - \mathbb{E}_t c_{t+n} + \frac{1}{2} \text{VAR}_t c_{t+n} + n \log(\beta)$$

$$c_t = -n y_t^{(n)} + \mathbb{E}_t c_{t+n} - \frac{1}{2} \text{VAR}_t c_{t+n} - n \log(\beta) \quad (1)$$

Time t consumption is thus inversely related to yields and future income uncertainty but positively related to future income.

A.3 Yield and the Path of Interest Rates

Now we substitute a modified Euler equation into Equation (1) to make yield a function of interest rates and interest volatility instead of consumption. This shows that bond yields increase with increases in uncertainty about the expected path of interest rates.

Writing the Euler equation for the one-period bond in a slightly different way puts it in terms consumption and the interest rate:

$$1 = \mathbb{E}_t \left\{ \beta \frac{C_t}{C_{t+1}} R_t \right\}$$

$$\frac{1}{C_t} = \mathbb{E}_t \left\{ \beta \frac{1}{C_{t+1}} R_t \right\}$$

Note how this generalizes:

$$\frac{1}{C_{t+1}} = \mathbb{E}_{t+1} \left\{ \beta \frac{1}{C_{t+2}} R_{t+1} \right\}$$

Using substitution, this Euler equation can be modified as follows:

$$\frac{1}{C_t} = \mathbb{E}_t \left\{ \beta R_t \mathbb{E}_{t+1} \left\{ \beta \frac{1}{C_{t+2}} R_{t+1} \right\} \right\}$$

$$\frac{1}{C_t} = \mathbb{E}_t \left\{ \mathbb{E}_{t+1} \left\{ \beta^2 \frac{1}{C_{t+2}} R_t R_{t+1} \right\} \right\}$$

$$\frac{1}{C_t} = \mathbb{E}_t \left\{ \beta^n \frac{1}{C_{t+n}} \prod_{i=0}^{n-1} R_{t+i} \right\}$$

$$\frac{1}{\beta^n C_t} = \mathbb{E}_t \left\{ \frac{1}{C_{t+n}} \prod_{i=0}^{n-1} R_{t+i} \right\}$$

Define $r_t = \log(R_t)$, and set up the equation for the Taylor series approximations:

$$\frac{1}{\beta^n C_t} = \mathbb{E}_t \left\{ \exp \left(-\log(C_{t+n}) + \sum_{i=0}^{n-1} \log(R_{t+i}) \right) \right\}$$

$$\frac{1}{\beta^n C_t} = \mathbb{E}_t \left\{ \exp \left(-c_{t+n} + \sum_{i=0}^{n-1} r_{t+i} \right) \right\}$$

$$-n \log(\beta) - c_t = \log \left(\mathbb{E}_t \left\{ \exp \left(-c_{t+n} + \sum_{i=0}^{n-1} r_{t+i} \right) \right\} \right)$$

$$-n \log(\beta) - c_t = \log \left(\mathbb{E}_t \left\{ \exp(-c_{t+n}) \exp \left(\sum_{i=0}^{n-1} r_{t+i} \right) \right\} \right)$$

Perform two Taylor series approximations for both $\exp(\cdot)$ terms under the expectations operator. Then distribute:

$$-n \log(\beta) - c_t = \log \left(\mathbb{E}_t \left\{ \left(1 - c_{t+n} + \frac{1}{2} c_{t+n}^2 \right) \left(1 + \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \left(\sum_{i=0}^{n-1} r_{t+i} \right)^2 \right) \right\} \right)$$

$$-n \log(\beta) - c_t = \log \left(1 + \mathbb{E}_t \left\{ -c_{t+n} + \frac{1}{2} c_{t+n}^2 + \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \left(\sum_{i=0}^{n-1} r_{t+i} \right)^2 - c_{t+n} \sum_{i=0}^{n-1} r_{t+i} \right\} \right)$$

Take another Taylor series approximation:

$$\begin{aligned} -n \log(\beta) - c_t &= -\mathbb{E}_t c_{t+n} + \frac{1}{2} \mathbb{E}_t c_{t+n}^2 + \mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \mathbb{E}_t \left(\sum_{i=0}^{n-1} r_{t+i} \right)^2 - \mathbb{E}_t c_{t+n} \sum_{i=0}^{n-1} r_{t+i} \\ &\quad - \frac{1}{2} \left[(\mathbb{E}_t c_{t+n})^2 + \left(\mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} \right)^2 - 2 \mathbb{E}_t c_{t+n} \mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} \right] \end{aligned}$$

Use the definitions of variance and covariance to simplify. Solve for c_t :

$$-n \log(\beta) - c_t = -\mathbb{E}_t c_{t+n} + \frac{1}{2} \text{VAR}_t c_{t+n} + \mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} - \text{COV}_t c_{t+n} \sum_{i=0}^{n-1} r_{t+i}$$

$$c_t = \mathbb{E}_t c_{t+n} - \frac{1}{2} \text{VAR}_t c_{t+n} - \mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} - \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} + \text{COV}_t c_{t+n} \sum_{i=0}^{n-1} r_{t+i} - n \log(\beta)$$

Subtract Equation (1) from this. Then solve for yield:

$$\begin{aligned} 0 &= -ny_t^{(n)} + \mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} - \text{COV}_t c_{t+n} \sum_{i=0}^{n-1} r_{t+i} \\ ny_t^{(n)} &= \mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} - \text{COV}_t c_{t+n} \sum_{i=0}^{n-1} r_{t+i} \\ y_t^{(n)} &= \frac{1}{n} \left[\mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} - \text{COV}_t c_{t+n} \sum_{i=0}^{n-1} r_{t+i} \right] \end{aligned} \quad (2)$$

While its clear that bond yields depend on the future path of interest rates, this shows that the volatility of the path also contributes.

A.4 Term Premium

We now find the risk-neutral bond yield and subtract it from the yield in Equation (2) to get a measure of the term premium. This shows that term premium is almost completely determined by the uncertainty about the path of interest rates.

Start by pricing the following risk-neutral bonds:

$$q_t^{(n)} = \frac{1}{R_t} \mathbb{E}_t q_{t+1}^{(n-1)}$$

$$q_{t+1}^{(n-1)} = \frac{1}{R_{t+1}} \mathbb{E}_{t+1} q_{t+2}^{(n-1)}$$

Use substitution to make the risk-neutral bond price a function of the path of rates:

$$q_t^{(n)} = \frac{1}{R_t} \mathbb{E}_t \left\{ \frac{1}{R_{t+1}} \mathbb{E}_{t+1} q_{t+2}^{(n-1)} \right\}$$

$$q_t^{(n)} = \mathbb{E}_t \left\{ \mathbb{E}_{t+1} \left\{ \frac{1}{R_t} \frac{1}{R_{t+1}} q_{t+2}^{(n-1)} \right\} \right\}$$

$$q_t^{(n)} = \mathbb{E}_t \left\{ \prod_{i=0}^{n-1} \frac{1}{R_{t+i}} \right\}$$

$$q_t^{(n)} = \mathbb{E}_t \left\{ \exp \left(- \sum_{i=0}^{n-1} \log(R_{t+i}) \right) \right\}$$

Recall $r_t = \log(R_t)$:

$$\log(q_t^{(n)}) = \log \left(\mathbb{E}_t \left\{ \exp \left(- \sum_{i=0}^{n-1} r_{t+i} \right) \right\} \right)$$

Perform two Taylor series approximations:

$$\log(q_t^{(n)}) = \log \left(1 + \mathbb{E}_t \left\{ - \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \left(\sum_{i=0}^{n-1} r_{t+i} \right)^2 \right\} \right)$$

$$\log(q_t^{(n)}) = -\mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \mathbb{E}_t \left(\sum_{i=0}^{n-1} r_{t+i} \right)^2 - \frac{1}{2} \left(\mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} \right)^2$$

Use the definition of variance to simplify. Then solve for the risk-neutral yield:

$$\begin{aligned}\log(q_t^{(n)}) &= -\mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} + \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} \\ n\hat{y}_t^{(n)} &= \mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} - \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} \\ \hat{y}_t^{(n)} &= \frac{1}{n} \left[\mathbb{E}_t \sum_{i=0}^{n-1} r_{t+i} - \frac{1}{2} \text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} \right]\end{aligned}$$

It is standard to write the term premium as the difference between a bond's yield and the yield of its risk-neutral counterpart, so subtract the risk-neutral yield above from the yield in Equation (2):

$$TP_t^{(n)} \triangleq y_t^{(n)} - \hat{y}_t^{(n)} = \frac{1}{n} \left[\text{VAR}_t \sum_{i=0}^{n-1} r_{t+i} - \text{COV}_t c_{t+n} \sum_{i=0}^{n-1} r_{t+i} \right] \quad (3)$$

All else equal, term premia rise with increases in the expected volatility of the path of rates.

A.5 Covariance Term

In the main text, the covariance term in Equations (2) and (3) is omitted. We show why by further decomposing the term premium.

The term premium can be rewritten using the definition of correlation:

$$\text{SD}_t X = \sqrt{\text{VAR}_t X};$$

$$\text{COV}_t(X, Y) = \text{COR}_t(X, Y) \text{SD}_t(X) \text{SD}_t(Y)$$

$$nTP_t^{(n)} = \text{SD}_t \sum_{i=0}^{n-1} r_{t+i} \left(\text{SD}_t \sum_{i=0}^{n-1} r_{t+i} - \text{COR}_t \left(c_{t+n}, \sum_{i=0}^{n-1} r_{t+i} \right) \text{SD}_t c_{t+n} \right)$$

Notice that decreasing volatility of the path of rates will only decrease term premium if

$$\begin{aligned}\text{COR}_t \left(c_{t+n}, \sum_{i=0}^{n-1} r_{t+i} \right) \text{SD}_t c_{t+n} < 0 \text{ or} \\ \text{SD}_t \sum_{i=0}^{n-1} r_{t+i} > \text{COR}_t \left(c_{t+n}, \sum_{i=0}^{n-1} r_{t+i} \right) \text{SD}_t c_{t+n}.\end{aligned}$$

Either case can be argued. Consider the first case: If the path of rates is lowered by a policy shock, it is likely to increase future consumption—making the correlation negative. Because we analyze policy shocks exclusively, this might be fairly reasonable. The argument for the second might be more convincing. As n increases, c_{t+n} approaches permanent income, which is likely to vary little with the path of rates. Thus, regardless of sign, $\text{COR}_t(c_{t+n}, \sum_{i=0}^{n-1} r_{t+i})$ is likely to be very small.