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# The Equilibrium Term Structure of Equity and Interest Rates

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## Abstract

We develop an equilibrium asset pricing model with Epstein-Zin recursive preferences that accounts for major stylized facts of the term structure of bond and equity risk premia. While the term structure of bond risk premia tends to be upward-sloping on average, the term structure of equity risk premia is known to be downward-sloping. The equilibrium asset pricing model with long-run consumption risks has difficulty in matching these stylized facts simultaneously. The standard calibration of these models follows Bansal and Yaron (2004) in which agents prefer the early resolution of uncertainty and have the inter-temporal elasticity of substitution greater than one; this calibration implies an upward-sloping term structure of equity risk premia and a downward-sloping term structure of real bond risk premia. Although it is shown that the standard model can match a downward-sloping term structure of equity risk premia by amplifying the short-run risk of dividend growth, it does not fully reconcile the model with empirical evidence implying an upward-sloping average yield curve and a downward-sloping term structure of Sharpe ratios of dividend strips. We extend a standard model in two dimensions. First, we incorporate time-varying market prices of risks by allowing marginal utility of consumption to be nonlinearly dependent on risk factors. Second, we endogenously determine expected cash flows and expected inflation as potentially nonlinear functions of risk factors. With these extensions, our model can match the average slope of both bond and equity risk premia together with the term structure of Sharpe ratios of dividend strips. At the same time, the model generates the behavior of the aggregate stock market return in line with the data.

Key Words: Long-run consumption risks; Term Structure; Interest Rates; Equity Risk Premia. JEL Classification: E43; G12.

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<sup>\*</sup>The views expressed herein are solely those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of Kansas City, the Federal Reserve System.

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# 1 Introduction

While recent advances in consumption-based asset pricing models provide solutions for several stylized facts once considered puzzling (e.g., high equity premium, low risk-free rate), explaining both the dynamics and the term structure of risk premia in an equilibrium asset pricing remains challenging. In particular, empirical studies indicate that the average shape of the term structure varies across different asset markets. While the term structure of bond risk premia is upward sloping, the term structure of equity risk premia inferred from cross-sectional stock market data or derivatives markets is found to be downward-sloping (van Binsbergen et al. (2012b), Weber (2016), etc.). Figure 1 provides evidence from the U.S. data that the average shape of the real and nominal bond yield curves is upward-sloping.

Several papers have provided a risk-based explanation using no-arbitrage dynamic asset pricing models including Ang and Ulrich (2012), Campbell, Sunderam, and Viceira (2013), Koijen et al. (2016), Lettau and Wachter (2011) among others. However, unlike an equilibrium asset pricing model, these models do not link the pricing kernel with investors' preferences. While convenient, this approach may not be able to provide clear economic interpretation of risk pricing. Sometimes, the assumed correlation of various risk factors to match stylized facts is not easily reconcilable with economic theories. For example, Lettau and Wachter (2011) assume that the real interest rate is negatively correlated with cash flow risk although this assumption implies a negative intertemporal elasticity of substitution when the real interest rate is determined in an equilibrium asset pricing model. Most papers based on the equilibrium approach focus only on term structure of bond risk premia or term structure of equity risk premia, respectively, but do not necessarily consider both simultaneously.<sup>1</sup>

As pointed out by Beeler and Campbell (2012), the standard calibration of the equilibrium asset pricing model based on long-run consumption risks typically implies a steeply downward-sloping term structure of real bond risk premia. The main reason is that investors prefer the early resolution of uncertainty in the standard calibration of preference parameters in the model and are willing to accept the lower yield of the long-term real bond that eliminates the uncertainty of the distant future. For the same reason, investors may demand a higher risk premium for a long-term dividend strip

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<sup>1</sup>Marfè (2016) is an exception. However, the paper does not address the term structure of nominal bond risk premia nor generate a downward sloping term structure of Sharpe ratios of dividend strips as we do in this paper.

than a short-run dividend strip when the dividend growth is sufficiently sensitive to consumption risk through the leverage channel because the uncertainty regarding the payoff of the long-run dividend strip is resolved later. Hence, it is challenging to match the term structure of real bond and equity risk premia simultaneously using the standard calibration of long run risks models.

Recent papers extended the standard long run risks model in various dimensions to explain stylized facts of the term structure of risk premia. For example, many papers introduce the nominal risk factor interacting with the long run consumption risk to explain an upward-sloping term structure of nominal bond risk premia (Bansal and Shaliastovich (2013), van Binsbergen et al. (2012a), Campbell, Pflueger, and Viceira (2015), Creal and Wu (2015), Doh (2013), Eraker and Shaliastovich and Wang (2015), Gallmeyer et al. (2008), Kung (2015), Piazzesi and Schneider (2007), and Song (2014) among others). These models match the average shape of the nominal bond yield curve by highlighting long-term nominal bond as a bad hedge for long run consumption risk or long run inflation uncertainty. Papers that focus on the term structure of equity risk premia include Belo et al. (2015), van Binsbergen and Koijen (2015), Croce et al. (2015), Hasler and Marfè (2016), and Marfè (2016) among others. They do match the downward-sloping term structure of equity risk premia by making dividend growth process more procyclical in the short-run, driving up the risk premium of the near-term dividend strip.<sup>2</sup>

In this paper we develop an equilibrium asset pricing model with Epstein-Zin recursive preferences and long-run consumption risks that can match the term structure of bond and equity risk premia simultaneously while maintaining properties such as the preference for the early resolution of uncertainty assumed in the standard calibration of long-run risks models. We do this by using a quadratic approximation for the log of wealth/consumption ratio as in Le and Singleton (2010) and Doh and Wu (2015) to endogenously generate time-varying market prices of risks that decrease the insurance value of long-run real and nominal bonds under persistent consumption or inflation shocks. This modification helps our model generate an upward sloping term structure of bond

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<sup>2</sup>Curatola (2015) and Lopez et al. (2015) modify Campbell and Cochrane (1999) habit-based models to explain the downward-sloping term structure of equity risk premia. Curatola (2015) combines loss-aversion with habit-based preferences and presents a downward-sloping term structure of equity risk premia because investors dislike the risk of falling below the habit level, which is much greater in the short-run and in the long-run. In Lopez et al. (2015), dividends (profits) are more procyclical in the short-run because marginal costs are countercyclical due to nominal rigidities. The habit-based preferences amplifies the risk pricing of the procyclical dividend in the short-run, generating a downward-sloping term structure of equity risk premia.

risk premia that is consistent with empirical evidence. To account for the downward sloping term structure of equity risk premia, we reverse engineer the expected dividend process from asset pricing restrictions. The recovered dividend growth process implies a time-varying leverage ratio that amplifies the short-run risk of dividend streams. In this respect, our approach is comparable to Belo et al. (2015), who tweak the dividend growth process in an otherwise standard long-run risks model *à la* Bansal and Yaron (2004) to generate the downward sloping term structure of equity risk premia. However, unlike Belo et al. (2015), we generate an upward-sloping term structure of (real and nominal) bond risk premia at the same time. In addition, we obtain the downward-sloping term structure of Sharpe ratios of dividend strips while matching the level of the aggregate stock market excess return. Interestingly, we match all these stylized facts while keeping the standard calibration of preference parameters implying investors' preferences for the early resolution of uncertainty. In a simple version of the long-run risks model, this calibration of preference parameters posed a challenge to match the stylized facts of the average shape of term structure of bond and equity risk premia. We find that the time-varying market price of risk originating from the nonlinear approximation of the log of wealth/consumption ratio improves the model's implications substantially in these dimensions by amplifying the volatility of the market price of risk in the short-run.

While our model is also broadly consistent with volatility and correlation of cash flows and inflation, it shares a shortcoming common in the long-run risks literature in terms of overstating the predictability of cash flows like dividend growth. Schorfheide et al. (2016) show that introducing multiple volatility shocks and allowing measurement errors in consumption growth alleviates this issue. For simplicity, we do not have stochastic volatility shocks in the model and generate time-varying volatility of cash flows and asset prices solely from the nonlinear dynamics of risk factors affecting expected cash flows. Incorporating volatility shocks is likely to improve the model's implications for cash flow dynamics at the expense of reducing the tractability.

The paper is organized as follows. Section 2 describes the equilibrium long-run risks model that we use to analyze the term structure of equity and interest rates. We illustrate how to obtain time-varying market price of risk from the nonlinear approximation of the wealth-consumption ratio. Section 3 discusses why the existing long-run risks model has difficulty in matching the stylized facts of the term structure of risk premia across different asset markets and how our model can overcome this difficulty. Section 4 calibrates model parameters to match the stylized facts of term structure of risk premia as well as prominent moments for aggregate cash flows and inflation. Section

5 concludes with a discussion of the way to improve the model's implications further regarding moments of the aggregate cash flows and inflation.

## 2 The Quadratic Asset Pricing Model

### 2.1 State Variables and Cash Flows

We assume that all risk factors relevant for asset pricing are summarized by a  $2 \times 1$  Markovian vector  $X_t$  that follows an affine process:

$$\Delta X_{t+1} = X_{t+1} - X_t = \Phi X_t + \Sigma_x \varepsilon_{x,t+1}, \quad (1)$$

where  $\varepsilon_{x,t+1}$  is a  $2 \times 1$  i.i.d. multivariate standard normal random shock and  $g_x(X_t)$  is an affine function of  $X_t$  under the restrictions such that  $X_t$  is stationary and ergodic.  $\Sigma_x$  is a  $2 \times 2$  matrix.<sup>3</sup> We denote  $\Omega_x = \Sigma_x \Sigma_x'$  and assume  $\Omega_x$  is invertible.

We consider two real cash flows, aggregate consumption and aggregate stock market dividend. We assume the growth rates of real cash flows and the rate of inflation are given by:

$$\Delta c_{t+1} = g_c(X_t) + \sigma_c \varepsilon_{c,t+1}, \quad (2)$$

$$\Delta d_{t+1} = g_d(X_t) + \rho_{cd} \varepsilon_{c,t+1} + \sigma_d \varepsilon_{d,t+1}, \quad (3)$$

$$\pi_{t+1} = g_\pi(X_t) + \sigma_\pi \varepsilon_{\pi,t+1}. \quad (4)$$

In principle, we can not only allow correlations among shocks to the risk factors affecting expected cash flows and expected inflation but also correlations among risk factors and transitory shocks to real cash flows and inflation.<sup>4</sup> However, to distinguish persistent

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<sup>3</sup>Unlike Doh and Wu (2015), we specify the dynamics of risk factors under the real world probability measure ( $\mathbb{P}$ ) not the risk-neutral probability measure ( $\mathbb{Q}$ ). Since asset pricing restrictions in Doh and Wu (2015) involve only a one-period ahead stochastic discount factor, the change of probability measure from ( $\mathbb{Q}$ ) to ( $\mathbb{P}$ ) is simple. However, the term structure model involves a multi-period ahead stochastic discount factor and inducing a similar probability measure change for the dynamics of risk factors is complicated not least because the risk-adjustment depends on the horizon of the stochastic discount factor.

<sup>4</sup>Some of these correlations are found to be useful for matching stylized facts in term structure of bond and equity risk premia. In Lettau and Watcher (2007, 2011), shocks to  $\Delta d_{t+1}$  are negatively correlated with shocks to  $X_{t+1}$  (or expected dividend growth). An unexpected higher dividend growth predicts lower future dividend growth and this channel is useful for generating a downward sloping term structure of equity risk premia that is consistent with the U.S. data. In Piazzesi and Schneider (2007), shocks to

risk factors from transitory shocks, we assume the covariance matrix is block-diagonal. We let

$$Y_{t+1} = \begin{pmatrix} \Delta X_{t+1} \\ \Delta c_{t+1} \\ \Delta d_{t+1} \\ \pi_{t+1} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Omega_x & 0 & 0 & 0 \\ 0 & \sigma_c^2 & \rho_{cd}^2 \sigma_c^2 & 0 \\ 0 & \rho_{cd}^2 \sigma_c^2 & \rho_{cd}^2 \sigma_c^2 + \sigma_d^2 & 0 \\ 0 & 0 & 0 & \sigma_\pi^2 \end{pmatrix}$$

At this point, we do not pin down the functional forms of  $g_c(X_t), g_d(X_t), g_\pi(X_t)$ . These will be determined by asset pricing restrictions that we will describe later.

## 2.2 Investor Preferences

To use equilibrium asset pricing restrictions to pin down functional forms of expected cash flows and expected inflation, we need to introduce the stochastic discount factor that investors use to price different assets. We derive the stochastic discount factor in the model by assuming that investors are endowed with the following Epstein-Zin (1989) recursive preferences in which the utility function is defined by:

$$U_t = [(1 - \delta)C_t^{1-1/\psi} + \delta(E_t[U_{t+1}^{1-\gamma}])^{1/\theta}]^{1-1/\psi}. \quad (5)$$

Here,  $0 < \delta < 1$  is the time discount factor,  $\gamma > 0$  is the parameter of risk-aversion,  $\psi > 0$  is the parameter of intertemporal elasticity of substitution and  $\theta = \frac{1-\gamma}{1-1/\psi}$ . Epstein and Zin (1991) derive the log stochastic discount factor ( $m_{t+1}$ ) as follows:

$$-m_{t+1} = -\theta \ln \delta + \frac{\theta}{\psi} \Delta c_{t+1} - (\theta - 1)r_{c,t+1}. \quad (6)$$

Here,  $r_{c,t+1}$  is the *real* return on the asset that pays aggregate consumption each period as its dividend. Investors prefer the early (late) resolution of uncertainty if  $\gamma > (<)\frac{1}{\psi}$ . Notice that this property of preference parameters is independent of how  $r_{c,t+1}$  is approximated as a function of risk factors.

The above stochastic discount factor is used to price assets whose payoffs are real. To price assets with nominal payoffs, we use the following log nominal stochastic discount

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$\pi_{t+1}$  are negatively correlated with shocks to  $X_{t+1}$  (or expected consumption growth), implying that unexpectedly higher inflation signals bad times ahead. This channel helps the model to generate an upward-sloping term structure of nominal bond risk premia that is supported by the U.S. data.

factor ( $m_{t+1}^n$ ).

$$m_{t+1}^n = m_{t+1} - \pi_{t+1}. \quad (7)$$

### 2.3 Expected Cash Flows and Expected Inflation

Instead of making econometric assumptions about expected cash flows and expected inflation, we directly postulate the log of wealth/consumption ratio and the log of price/dividend ratio of the one-period ahead aggregate dividend strip as quadratic functions of the state vector  $X_t$ . We take coefficients on risk factors in the wealth/consumption ratio and the dividend strip price as free parameters and use equilibrium asset pricing restrictions to reverse engineer expected cash flows given investor's preferences. Assuming a Taylor-rule type of monetary policy, we can also solve for expected inflation. Doh and Wu (2015) show that this reverse-engineering can nest the standard approach in the long-run risks literature that market prices of risks are endogenously determined from asset pricing restrictions while expected cash flows follow exogenously specified stochastic processes. In this paper, we show that this result holds with expected inflation as well as expected cash flows.

The advantage of our approach is the tractability in deriving equilibrium asset prices when market prices of risks can be time-varying, for example, as affine functions of risk factors. In this case, we have to solve complicated quadratic equations to determine market prices of risks parameters if we follow the standard approach. On the other hand, in our reverse-engineering approach, coefficients of expected cash flows and expected inflation on risk factors are uniquely determined by other model parameters.<sup>5</sup> Hence, our approach generates analytically tractable forms for equilibrium asset prices even when market prices of risks are time-varying. The tractability is important because they make transparent the underlying sources of variations in equity and bond risk premia over time *and* across maturities.

The disadvantage of our approach compared to the standard approach, however, is that the underlying risk factors cannot be directly interpreted as expected cash flows or inflation other than factors determining these expectations. We look at responses of real cash flows and inflation to a one standard deviation shock to each risk factor to trace

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<sup>5</sup>The appendix illustrates a complication in deriving affine market prices of risks under the standard approach.

out economic implications of risk factors.

### 2.3.1 Asset Prices

To back out expected cash flows and expected inflation from asset pricing restrictions, we first need to specify functional forms of asset prices. Since we have two measures of real cash flows (consumption growth and dividend growth) and one inflation measure, we introduce three asset prices.

#### Consumption claim

First, we model the price of consumption claim. The total wealth can be regarded as the price of consumption claim. We postulate that the log of wealth/consumption ratio is given by a quadratic function of the risk factors:

$$z_{c,t} = f_c(X_t) = \lambda_{c,0} + \lambda'_{c,1}X_t + \frac{1}{2}X'_tH_cX_t. \quad (8)$$

Using Campbell-Shiller (1989) log-linear approximation, the one period holding return on the consumption claim can be written as:

$$r_{c,t+1} = k_{c,0} - k_{c,1}z_{c,t} + k_{c,2}\Delta z_{c,t+1} + \Delta c_{t+1}, \quad (9)$$

where  $k_{c,1} = 1 - k_{c,2}$ ,  $k_{c,2} = \frac{e^{\bar{z}_c}}{1+e^{\bar{z}_c}}$  and  $k_{c,0} = \ln(1 + e^{\bar{z}_c}) - \frac{\bar{z}_c e^{\bar{z}_c}}{1+e^{\bar{z}_c}}$ .  $\bar{z}_c$  is the steady-state value of wealth/consumption ratio  $z_{c,t}$  and it is given by  $\lambda_{c,0}$ .

As in Le and Singleton (2010), we will use the following approximation for the percentage change of asset prices throughout the paper:<sup>6</sup>

$$\Delta f(X_{t+1}) \approx \Gamma(X_t)' \Delta X_{t+1}$$

where  $\Gamma(X_t) = \frac{\partial f(X_t)}{\partial X_t}$ . Applied to the log of wealth consumption ratio,  $\Gamma_c(X_t) = \lambda_{c,1} + \frac{1}{2}(H_c + H'_c)X_t = \lambda_{c,1} + \bar{H}_c X_t$ .

Hence the real return on the consumption claim can be obtained approximately as:

$$r_{c,t+1} \approx k_{c,0} - k_{c,1}f_c(X_t) + k_{c,2}\Gamma_c(X_t)' \Delta X_{t+1} + \Delta c_{t+1}. \quad (10)$$

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<sup>6</sup>As long as  $\Omega_x$  is sufficiently small relative to quadratic coefficients in log price/cash flow ratios, which is indeed the case, this approximation is fairly accurate.

Market price of risks attached to the consumption claim  $h_c(X_t)$  can be calculated as follows:

$$r_{c,t+1} - E_t(r_{c,t+1}) = h_c(X_t)'(Y_{t+1} - E_t(Y_{t+1})), \quad (11)$$

where  $h_c(X_t) = [\kappa_{c,2}\Gamma_c(X_t), 1, 0, 0]'$ .

### Dividend Strip Claim

Next we specify the price of one-period ahead dividend strip. Consider a dividend stream  $\{D_t\}_{t=1}^{\infty}$ . We let  $z_{d,t} = \ln(P_{d,t}/D_t)$  denote the log price/dividend ratio of the dividend claim. We further let  $P_{n,t}^d$  to be the time  $t$  real price of the dividend strip of maturity  $n$  and define  $z_{n,t}^d = \ln(P_{n,t}^d/D_t)$ . By construction,

$$z_{d,t} = \ln\left(\sum_{n=1}^{\infty} e^{z_{n,t}^d}\right).$$

We assume that the log price/dividend ratio of the one-period ahead dividend strip is a quadratic function of risk factors:

$$z_{d,t}^1 = f_d^1(X_t) = \lambda_{d,0} + \lambda'_{d,1}X_t + \frac{1}{2}X_t'H_dX_t. \quad (12)$$

The real return on the one-period ahead dividend strip is then given by:

$$r_{d,t+1}^1 = \Delta d_{t+1} - f_d^1(X_t). \quad (13)$$

### Monetary Policy Rule and the Short-term Nominal Interest Rate

Finally, we assume the short-term nominal interest rate ( $i_{1,t}$ ) is determined by a forward-looking Taylor-rule type of monetary policy as follows:

$$i_{1,t} = \bar{r} + \alpha_c g_c(X_t) + \alpha_\pi g_\pi(X_t) + i'_m X_t. \quad (14)$$

Notice that  $g_c(X_t)$  and  $g_\pi(X_t)$  are expected consumption growth and expected inflation and  $\alpha_c$  and  $\alpha_\pi$  are coefficients describing policy response to expected consumption growth and expected inflation.  $i_m = (0, 1)'$  is a  $2 \times 1$  vector that picks up the second element of  $X_t$ .<sup>7</sup> We can derive from the price of the one-period ahead (default-risk free) nominal

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<sup>7</sup>Under certain restrictions on model parameters that will be discussed later,  $i'_m X_t$  can be interpreted as a nominal shock affecting expected inflation.

bond ( $P_{1,t}$ ) from  $i_{1,t}$  by using the fact that  $P_{1,t} = e^{-i_{1,t}}$ .

Since we assume quadratic functions for all the three asset prices, the resulting expressions for expected cash flows and expected inflation are also quadratic functions.

$$g_c(X_t) = g_{c,0} + g'_{c,1}X_t + \frac{1}{2}X_t'W_cX_t, \quad (15)$$

$$g_d(X_t) = g_{d,0} + g'_{d,1}X_t + \frac{1}{2}X_t'W_dX_t, \quad (16)$$

$$g_\pi(X_t) = g_{\pi,0} + g'_{\pi,1}X_t + \frac{1}{2}X_t'W_\pi X_t. \quad (17)$$

Coefficients in these functions are later determined using asset pricing restrictions.

### 2.3.2 Stochastic Discount Factor

Substituting the approximation for  $r_{c,t+1}$  into (6), the stochastic discount factor becomes:

$$\begin{aligned} -m_{t+1} = & -\theta \ln \delta - (\theta - 1)k_{c,0} + (\theta - 1)k_{c,1}f_c(X_t) \\ & - (\theta - 1)k_{c,2}\Gamma_c(X_t)'\Delta X_{t+1} + \gamma\Delta c_{t+1} \end{aligned} \quad (18)$$

To simplifying notations, we let:

$$\begin{aligned} m_0 &= -\theta \ln \delta - (\theta - 1)[k_{c,0} - k_{c,1}\lambda_{c,0} + \gamma g_{c,0}], \\ m_1(X_t) &= (\theta - 1)(k_{c,1}\lambda'_{c,1}X_t + k_{c,1}\frac{1}{2}X_t'H_cX_t - k_{c,2}\Gamma_c(X_t)'\Phi X_t) + \gamma(g'_{c,1}X_t + \frac{1}{2}X_t'W_cX_t), \\ h(X_t) &= [(1 - \theta)k_{c,2}\Gamma_c(X_t)', \gamma, 0, 0]'. \end{aligned}$$

Now we can decompose the log stochastic discount factor into the expected component and the unexpected component:

$$-m_{t+1} = m_0 + m_1(X_t) + h(X_t)'(Y_{t+1} - E_t(Y_{t+1})). \quad (19)$$

We refer to the  $5 \times 1$  vector,  $h(X_t)$ , as the market price of risk below.

### 2.3.3 Determining Expected Cash Flows

In equilibrium, the representative agent should be indifferent between the additional investment in asset  $i$  to achieve a higher future consumption with the future return of  $R_{i,t+1}$  and the additional consumption in the current period. This implies that every asset should provide the same expected return when risk is adjusted by the stochastic discount factor as follows:

$$\frac{\partial U_t}{\partial C_t} = E_t\left(\frac{\partial U_t}{\partial C_{t+1}} R_{i,t+1}\right) \Rightarrow 1 = E_t(e^{m_{t+1}+r_{i,t+1}}). \quad (20)$$

If we apply the above equilibrium asset pricing restriction to three assets introduced in the previous section (consumption claim, one-period ahead dividend strip, and one-period nominal government bond), the following three equations are obtained.

$$E_t(e^{m_{t+1}+r_{c,t+1}}) = E_t(e^{m_{t+1}+\Delta d_{t+1}-z_{d,t}^1}) = E_t(e^{m_{t+1}^n+i_{1,t}}) = 1. \quad (21)$$

Using the equilibrium asset pricing restriction on  $r_{c,t+1}$  ( $E_t(m_{t+1}+r_{c,t+1})+\frac{1}{2}V_t(m_{t+1}+r_{c,t+1})=0$ ), we can determine coefficients in  $g_c(X_t)$  by matching coefficients in both sides of equations.

$$g_c(X_t) = g_{c,0} + g'_{c,1}X_t + \frac{1}{2}X_t'W_cX_t, \quad (22)$$

where

$$\begin{aligned} g_{c,0} &= -\frac{1}{1-1/\psi}[\ln \delta + k_{c,0} - k_{c,1}\lambda_{c,0}] - \frac{1-\gamma}{2}\sigma_c^2 - \frac{\theta^2 k_{c,2}^2}{2(1-\gamma)}\lambda'_{c,1}\Omega_x\lambda_{c,1}, \\ g_{c,1} &= \frac{1}{1-1/\psi}[k_{c,1}I - k_{c,2}\Phi]'\lambda_{c,1} - \frac{\theta^2 k_{c,2}^2}{1-\gamma}\bar{H}_c\Omega_x\lambda_{c,1}, \\ W_c &= \frac{1}{1-1/\psi}(k_{c,1}H_c - 2k_{c,2}\bar{H}_c\Phi) - \frac{\theta^2 k_{c,2}^2}{(1-\gamma)}\bar{H}_c\Omega_x\bar{H}_c, \quad \bar{H}_c = \frac{H_c + H'_c}{2}. \end{aligned}$$

Similarly, using the equilibrium asset pricing restriction on  $\Delta d_{t+1} - z_{d,t}^1$  ( $E_t(m_{t+1} + \Delta d_{t+1} - z_{d,t}^1) + \frac{1}{2}V_t(m_{t+1} + \Delta d_{t+1} - z_{d,t}^1) = 0$ ), we can derive coefficients of expected dividend growth.

$$g_d(X_t) = g_{d,0} + g'_{d,1}X_t + \frac{1}{2}X_t'W_dX_t, \quad (23)$$

where

$$g_{d,0} = \lambda_{d,0} + \gamma g_{c,0} - \theta \ln \delta - (\theta - 1)(k_{c,0} - k_{c,1} \lambda_{c,0}) - \frac{k_{c,2}^2 (\theta - 1)^2 \lambda'_{c,1} \Omega_x \lambda_{c,1} + (\rho_{cd} - \gamma)^2 \sigma_c^2 + \sigma_d^2}{2},$$

$$g_{d,1} = \lambda_{d,1} + \gamma g_{c,1} + (\theta - 1)(k_{c,1} I - k_{c,2} \Phi') \lambda_{c,1},$$

$$W_d = H_d + \gamma W_c + (\theta - 1) k_{c,1} H_c - 2(\theta - 1) k_{c,2} \bar{H}_c \Phi - k_{c,2}^2 (\theta - 1)^2 \bar{H}_c \Omega_x \bar{H}_c.$$

### 2.3.4 Determining Expected Inflation

To determine the coefficients in expected inflation, we follow Gallmeyer et al. (2008) and Song (2014) by equating the nominal short-term interest rate from the stochastic discount factor to the one implied by the monetary policy rule.

$$g_\pi(X_t) = g_{\pi,0} + g'_{\pi,1} X_t + \frac{1}{2} X_t' W_\pi X_t, \quad (24)$$

where

$$g_{\pi,0} = \frac{-\bar{r} - \theta \ln \delta + (1 - \theta)(k_{c,0} - k_{c,1} \lambda_{c,0}) + (\gamma - \alpha_c) g_{c,0} - \frac{\gamma^2 \sigma_c^2 + \sigma_\pi^2 + k_{c,2}^2 (\theta - 1)^2 \lambda'_{c,1} \Omega_x \lambda_{c,1}}{2}}{\alpha_\pi - 1},$$

$$g_{\pi,1} = \frac{1}{\alpha_\pi - 1} [(\theta - 1)((k_{c,1} I - k_{c,2} \Phi') \lambda_{c,1}) + (\gamma - \alpha_c) g_{c,1} - (\theta - 1)^2 k_{c,2}^2 \bar{H}_c \Omega_x \lambda_{c,1} - i_m],$$

$$W_\pi = \frac{1}{\alpha_\pi - 1} [(\theta - 1)(k_{c,1} H_c - 2k_{c,2} \bar{H}_c \Phi) + (\gamma - \alpha_c) W_c - (\theta - 1)^2 k_{c,2}^2 \bar{H}_c \Omega_x \bar{H}_c].$$

## 2.4 The Term Structure of Interest Rates and Bond Risk Premia

Let  $i_{n,t}$  and  $r_{n,t}$  be the  $n$ -period nominal and real interest rate respectively. Then we must have in equilibrium that:

$$e^{-ni_{n,t}} = e^{-it} E_t \left( e^{m_{i+1}^n - (n-1)i_{n-1,t+1}} \right), \quad (25)$$

$$e^{-nr_{n,t}} = e^{-rt} E_t \left( e^{m_{r+1}^n - (n-1)r_{n-1,t+1}} \right). \quad (26)$$

where the recursions start with initial conditions that  $i_{0,t} = 0$  and  $r_{0,t} = 0$ . We solve in closed-form for both  $i_{n,t}$  and  $r_{n,t}$  below.

#### 2.4.1 The Term Structure of Nominal Interest Rates

The nominal risk-free rate can be pinned down by the monetary policy given expected consumption growth and inflation:

$$i_{1,t} = A_{\$,1} + B'_{\$,1} X_t + \frac{1}{2} X'_t C_{\$,1} X_t, \quad (27)$$

where

$$A_{\$,1} = \bar{r} + \alpha_c g_{c,0} + \alpha_\pi g_{\pi,0},$$

$$B_{\$,1} = \alpha_c g_{c,1} + \alpha_\pi g_{\pi,1} + i_m,$$

$$C_{\$,1} = \alpha_c W_c + \alpha_\pi W_\pi.$$

Let  $P_{n,t}^\$$  be the nominal price at time  $t$  of a zero-coupon bond that pays \$1 at  $t + n$ . Equilibrium conditions then imply that:

$$P_{n,t}^\$ = e^{-ni_{n,t}} = E_t \left( e^{m_{t+1}^n} P_{n-1,t+1}^\$ \right)$$

. We can show that:<sup>8</sup>

$$P_{n,t}^\$ = e^{-A_{\$,n} - B'_{\$,n} X_t - \frac{1}{2} X'_t C_{\$,n} X_t}, \quad (28)$$

where for  $n \geq 2$ ,

$$A_{\$,n} = A_{\$,n-1} + A_{\$,1} + (\theta - 1)k_{c,2}\lambda'_{c,1}\Omega_x B_{\$,n-1} - \frac{1}{2}B'_{\$,n-1}\Omega_x B_{\$,n-1},$$

$$B_{\$,n} = B_{\$,1} + (\theta - 1)k_{c,2}(\bar{H}'_c \Omega_x B_{\$,n-1} + \bar{C}'_{\$,n-1} \Omega_x \lambda_{c,1}) + (I + \Phi - \Omega_x \bar{C}_{\$,n-1})' B_{\$,n-1},$$

$$C_{\$,n} = C_{\$,n-1} + C_{\$,1} + 2\bar{C}_{\$,n-1}\Phi + 2(\theta - 1)k_{c,2}(\bar{H}_c \Omega_x \bar{C}_{\$,n-1}) - \bar{C}_{\$,n-1}\Omega_x \bar{C}_{\$,n-1},$$

$$\bar{C}_{\$,n-1} = \frac{1}{2}(C_{\$,n-1} + C'_{\$,n-1}).$$

The yield for an  $n$ -period nominal bond is  $i_{n,t} = -\frac{\ln P_{n,t}^\$}{n} = \frac{A_{\$,n} + B'_{\$,n} X_t + \frac{1}{2} X'_t C_{\$,n} X_t}{n}$ .<sup>9</sup>

<sup>8</sup>We have used the same first-order Taylor approximation for a nonlinear function as above, i.e.  $\Delta f(X_{t+1}) \approx \frac{\partial f(X_t)}{\partial X_t} \Delta X_{t+1}$ , in order to derive the analytical solution for bond prices.

<sup>9</sup>To ensure that the nominal bond price converges in the long-run as  $n$  goes to  $\infty$ , we need to check if all the coefficients converge. The condition to guarantee finite values for  $\lim_{n \rightarrow \infty} C_{\$,n}$ ,  $\lim_{n \rightarrow \infty} B_{\$,n}$ ,  $\lim_{n \rightarrow \infty} A_{\$,n}$  is described in the appendix.

The one-period ahead expected excess return (risk premium) of an  $n$ -period zero-coupon nominal bond from  $t$  to  $t + 1$  can be obtained as:<sup>10</sup>

$$\begin{aligned}
rP_{n,t}^{\$} &= E_t[\ln P_{n-1,t+1}^{\$} - \ln P_{n,t}^{\$} - i_{1,t}]. \\
&\approx (A_{\$,n} - A_{\$,n-1} - A_{\$,1}) \text{ (Constant Term)} \\
&+ (B_{\$,n} - B_{\$,n-1}(I + \Phi) - B_{\$,1})' X_t \text{ (Linear Term)} \\
&+ \left( \frac{X_t'(C_{\$,n} - C_{\$,n-1} - C_{\$,1} - 2\bar{C}_{\$,n-1}\Phi) X_t}{2} \right) \text{ (Quadratic Term)}.
\end{aligned} \tag{29}$$

Notice that the expected excess return would be constant in the linear model because  $\bar{C}_{\$,n-1}$  and  $H_c$  are both equal to zero and  $E(X_t) = 0$ .

#### 2.4.2 The Term Structure of Real Interest Rates

From the Euler equation, the short-term real interest rate is obtained by

$$e^{-r_{1,t}} = E_t(e^{m_{t+1}}) \rightarrow r_{1,t} = -E_t(m_{t+1}) - \frac{V_t(m_{t+1})}{2} = A_1 + B_1' X_t + \frac{X_t' C_1 X_t}{2}. \tag{30}$$

We can show that

$$\begin{aligned}
A_1 &= -\theta \ln(\delta) + \gamma g_{c,0} + (1 - \theta)(k_{c,0} - k_{c,1}\lambda_{c,0}) - \frac{\gamma^2 \sigma_c^2 + (\theta - 1)^2 k_{c,2}^2 \lambda'_{c,1} \Omega_x \lambda_{c,1}}{2}, \\
B_1 &= \gamma g_{c,1} + (\theta - 1)(k_{c,1} - k_{c,2}\Phi')\lambda_{c,1} - (\theta - 1)^2 k_{c,2}^2 \bar{H}_c \Omega_x \lambda_{c,1}, \\
C_1 &= \gamma W_c + (\theta - 1)k_{c,1}\bar{H}_c - 2(\theta - 1)k_{c,2}\bar{H}_c\Phi - (\theta - 1)^2 k_{c,2}^2 \bar{H}_c \Omega_x \bar{H}_c.
\end{aligned}$$

Let  $P_{n,t}$  to be the real price at time  $t$  of a zero-coupon bond that pays one unit of consumption good at  $t + n$ . In equilibrium we must have:

$$P_{n,t} = e^{-nr_{n,t}} = E_t(P_{n-1,t+1}e^{m_{t+1}}).$$

Using the above no-arbitrage condition, we can derive prices of real zero-coupon bonds:

$$P_{n,t} = e^{-A_n - B_n' X_t - \frac{1}{2} X_t' C_n X_t}, \tag{31}$$

<sup>10</sup>We approximate  $\frac{E_t(X_{t+1}' C_{\$,n-1} X_{t+1})}{2}$  by  $\frac{X_t' C_{\$,n-1} X_t}{2} + X_t' \bar{C}_{\$,n-1} (\Phi X_t + \varepsilon_{x,t+1})$ .

where for  $n \geq 2$ ,

$$\begin{aligned} A_n &= A_{n-1} + A_1 + (\theta - 1)k_{c,2}\lambda'_{c,1}\Omega_x B_{n-1} - \frac{1}{2}B'_{n-1}\Omega_x B_{n-1}, \\ B_n &= (I + \Phi - \Omega_x(\bar{C}_{n-1} - (\theta - 1)k_{c,2}\bar{H}_c))'B_{n-1} + B_1 + (\theta - 1)k_{c,2}\bar{C}_{n-1}\Omega_x\lambda_{c,1}, \\ C_n &= C_{n-1} + C_1 + 2\bar{C}_{n-1}\Phi - \bar{C}_{n-1}\Omega_x\bar{C}_{n-1} + 2(\theta - 1)k_{c,2}\bar{H}_c\Omega_x\bar{C}_{n-1}, \quad \bar{C}_{n-1} = \frac{1}{2}(C_{n-1} + C'_{n-1}). \end{aligned}$$

The yield for an  $n$ -period real bond is  $r_{n,t} = -\frac{\ln P_{n,t}}{n} = \frac{A_n + B'_n X_t + \frac{1}{2}X'_t C_n X_t}{n}$ .

The one-period ahead expected excess return of a  $n$ -period zero-coupon real bond from  $t$  to  $t + 1$  can be obtained as:

$$\begin{aligned} r p_{n,t} &= E_t[\ln P_{n-1,t+1} - \ln P_{n,t} - r_{1,t}]. \\ &\approx (A_n - A_{n-1} - A_1) \text{ (Constant Term)} \\ &+ (B_n - B_{n-1}(I + \Phi) - B_1)' X_t \text{ (Linear Term)} \\ &+ \left( \frac{X'_t (C_n - C_{n-1} - C_1 - 2\bar{C}_{n-1}\Phi) X_t}{2} \right) \text{ (Quadratic Term)}. \end{aligned} \tag{32}$$

Again, in a linear model, the expected excess return would be constant because  $\bar{C}_{n-1}$  and  $H_c$  are both equal to zero and  $E(X_t) = 0$ .

## 2.5 The Term Structure of Dividend Strips and Equity Risk Premia

Recall that  $P_{n,t}^d$  is the *real* price at  $t$  of a claim to a dividend that will be paid  $n$  periods from today,  $D_{t+n}$ . It then follows that:

$$P_{n,t}^d = E_t \left( e^{\sum_{i=1}^n m_{t+i}} D_{t+n} \right). \tag{33}$$

Define  $z_{n,t}^d$  to be  $\log(P_{n,t}^d/D_t)$ , we can rewrite the pricing equation above recursively as:

$$e^{z_{n,t}^d} = E_t \left( e^{\Delta_{t+1} + m_{t+1} + z_{n-1,t+1}^d} \right), \tag{34}$$

We assume that  $z_{d,t}^1 = f_d^1(X_t) = \lambda_{d,0} + \lambda'_{d,1} X_t + \frac{1}{2} X'_t H_d X_t$ .

We can show that, for all  $n \geq 2$ ,

$$z_{n,t}^d = A_{d,n} + B'_{d,n} X_t + \frac{1}{2} X'_t C_{d,n} X_t \quad (35)$$

where

$$\begin{aligned} A_{d,n} &= A_{d,n-1} + \lambda_{d,0} + \frac{(B_{d,n-1} + 2(\theta - 1)k_{c,2}\lambda_{c,1})'\Omega_x B_{d,n-1}}{2}, \\ B_{d,n} &= \lambda_{d,1} + (I + \Phi + \Omega_x ((\theta - 1)k_{c,2}\bar{H}_c + \bar{C}_{d,n-1}))' B_{d,n-1} + (\theta - 1)k_{c,2}\bar{C}_{d,n-1}\Omega_x \lambda_{c,1}, \\ C_{d,n} &= H_d + C_{n-1}^d + 2\bar{C}_{n-1}^d \Phi + \bar{C}_{d,n-1}\Omega_x \bar{C}_{n-1}^d + 2(\theta - 1)k_{c,2}\bar{H}_c\Omega_x \bar{C}_{d,n-1} \\ \bar{C}_{d,n-1} &= \frac{1}{2}(C_{d,n-1} + C'_{d,n-1}). \end{aligned} \quad (36)$$

The expected excess return of the dividend strip can be obtained as:

$$\begin{aligned} rp_{n,t}^d &= E_t \left[ z_{d,t+1}^{n-1} - z_{d,t}^n + \Delta d_{t+1} - r_{1,t} \right], \\ &\approx (A_{d,n-1} - A_{d,n} + g_{d,0} - A_1) \text{ (Constant Term)} \\ &\quad + (B_{d,n-1}(I + \Phi) + g_{d,1} - B_{d,n} - B_1)' X_t \text{ (Linear Term)} \\ &\quad + \left( \frac{X'_t (C_{d,n-1} - C_{d,n} - C_1 + W_d + 2\bar{C}_{d,n-1}\Phi) X_t}{2} \right) \text{ (Quadratic Term)}. \end{aligned} \quad (37)$$

### 3 Comparison with the Standard Linear Approach

#### 3.1 Economic interpretation of risk factors

Our model does not directly provide the exact economic interpretation of risk factors. However, under certain restrictions on model parameters, we can obtain the straightforward economic interpretation for  $X_t$ . For instance, we can make  $x_{1,t}$  expected consumption growth and  $x_{2,t}$  expected inflation while allowing the dynamic correlation between two risk factors as assumed in Bansal and Shaliastovich (2013) and Song (2014). If we set all the elements in  $H_c$  and  $H_d$  to zero, expected consumption growth in this linear model is given by

$$E_t(\Delta c_{t+1}) = g_c(X_t) = g_{c,0} + g'_{c,1} X_t, \quad (38)$$

$$g_{c,1} = \frac{1}{1 - 1/\psi} (k_{c,1}I - k_{c,2}\Phi)' \lambda_{c,1}. \quad (39)$$

To provide the standard interpretation for  $X_t$ , we need to have  $g_{c,1} = (1, 0)'$ . This requirement imposes additional restrictions on market price of risk parameters  $\lambda_{c,1}$ .

$$\lambda_{c,11} = \frac{1 - 1/\psi}{k_{c,1} - k_{c,2}\phi_{11}}, \quad \lambda_{c,12} = \frac{k_{c,2}\phi_{12}(1 - 1/\psi)}{(k_{c,1} - k_{c,2}\phi_{22})(k_{c,1} - k_{c,2}\phi_{11})}. \quad (40)$$

With these restrictions,  $x_{1,t}$  becomes the time-varying component in expected consumption growth. Similarly, we can impose restrictions on monetary policy rule parameters to induce  $x_{2,t}$  as expected inflation. Recall that expected inflation in the linear model takes the following form.

$$E_t(\pi_{t+1}) = g_\pi(X_t) = g_{\pi,0} + g_{\pi,1}X_t, \quad g_{\pi,1} = \frac{1}{1 - \alpha_\pi}[(\alpha_c - 1/\psi)g_{c,1} + i_m]. \quad (41)$$

If we restrict  $\alpha_c$  to be  $1/\psi$ ,  $\frac{x_{2,t}}{1 - \alpha_\pi}$  becomes the time-varying component of expected inflation. In this case, the monetary policy rule is just the restatement of the Euler equation for the short-term nominal interest rate where  $X_t$  corresponds to  $[E_t(\Delta c_{t+1}) - E(\Delta c_{t+1}), (1 - \alpha_\pi)(E_t(\pi_{t+1}) - E(\pi_{t+1}))]'$ .

In addition, we can impose the following restrictions on  $\lambda_{d,1}$  to make the time-varying component of expected dividend growth a constant multiple of the time-varying component of expected consumption growth as in Bansal and Yaron (2004).

$$E_t(\Delta d_{t+1}) = g_d(X_t) = g_{d,0} + g_{d,1}X_t, \quad (42)$$

$$g_{d,1} = \lambda_{d,1} + \frac{g_{c,1}}{\psi} = [\lambda_{d,11} + 1/\psi, \lambda_{d,12}]'. \quad (43)$$

Therefore, if  $\lambda_{d,12}$  is equal to zero,  $g_{d,1}$  can be set to  $(\lambda_{d,11} + 1/\psi) \times g_{c,1} = q \times g_{c,1}$  where  $q$  can be interpreted as the leverage ratio.

### 3.2 Term structure implications

As pointed out by van Binsbergen et al. (2012b) and Croce et al. (2015), the standard calibration of the long-run risks model such as Bansal and Yaron (2004) implies the steeply upward-sloping term structure of equity risk premia that is at odds with the available data on cross-sectional stock returns and dividend strip prices inferred from equity derivatives. In the U.S. data, stocks with longer duration of cash flows (growth stocks) earn smaller excess market returns (risk premia) than those with shorter duration of cash flows (value stocks). Also, dividend strip prices inferred from equity derivative

markets imply that an investment strategy buying and holding long-run dividend strips earns smaller excess market returns than an investment strategy buying and holding short-run dividend strips. All these facts suggest the downward-sloping term structure of equity risk premia. Why does the standard calibration of the long-run risks model fail to match this fact? The main reason behind this failure is that investors prefer the early resolution of uncertainty under the standard calibration and hate to bear long-horizon cash flow risks, demanding higher risk premia for assets with greater exposures to long-run cash flow risks. Following Croce et al. (2015), this problem can be analytically shown by deriving the mean excess return on an  $n$ -period dividend strip in a simple one factor long-run risks model.

Suppose that consumption growth and dividend growth follow processes described below as in the one-factor long-run risks model in Bansal and Yaron (2004).

$$\Delta c_{t+1} = \mu_c + x_t + \epsilon_{c,t+1}, \quad x_{t+1} = \rho_x x_t + \epsilon_{x,t+1}, \quad \epsilon_{x,t+1} \sim \mathcal{N}(0, \sigma_x^2), \quad (44)$$

$$\Delta d_{t+1} = \mu_d + q x_t + \epsilon_{d,t+1}, \quad \epsilon_{c,t+1} \sim \mathcal{N}(0, 1), \quad \epsilon_{d,t+1} \sim \mathcal{N}(0, 1). \quad (45)$$

The mean excess return on an  $n$ -period dividend strip ( $rp d_n = E(\frac{P_{n-1,t+1}^d}{P_{n,t}^d} - e^{r_{1,t}})$ ) can be derived as follows.<sup>11</sup>

$$rp d_n = \left[ \frac{(q - 1/\psi)(1 - \rho_x^n)(\gamma - 1/\psi)k_{c,2}}{(1 - \rho_x)(1 - k_{c,2}\rho_x)} \right] \sigma_x^2. \quad (46)$$

Under the typical calibration, investors prefer the early resolution of uncertainty and have the intertemporal elasticity of substitution greater than 1, implying  $\gamma > 1 > \frac{1}{\psi}$ . In addition,  $\rho_x$  is a positive number slightly below 1 and the leverage ratio  $q$  is bigger than 1, implying  $q > \frac{1}{\psi}$ . Under this calibration,  $rp d_n$  is always positive and an increasing function of  $n$ . Hence, the term structure of equity risk premia is upward-sloping on average.

Quite interestingly, the same channel generates another counterfactual pattern for the term structure of real bond risk premia. Since investors extremely dislike the long-horizon cash flow risk, they are willing to pay a high price for insurance against such a risk. Therefore, the insurance value of the long-run real bond is higher than that of the short-run real bond, resulting in a downward-sloping term structure of real bond risk

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<sup>11</sup>The definition of the expected excess return here ignores the second-order term arising from Jensen's inequality.

premia. In the simple model described above, the mean excess return on an  $n$ -period real bond ( $rpr_n = E(\frac{P_{n-1,t+1}}{P_{n,t}} - e^{r_{1,t}})$ ) can be derived as follows:

$$rpr_n = -\left[\frac{(1 - \rho_x^n)(\gamma - 1/\psi)k_{c,2}}{\psi(1 - \rho_x)(1 - k_{c,2}\rho_x)}\right]\sigma_x^2. \quad (47)$$

Under the same standard calibration, the model now generates a downward-sloping term structure of real bond risk premia. Hence, in spite of the success in matching the average level of the overall equity risk premium and the real risk-free rate, the standard calibration of the long-run risks model in Bansal and Yaron (2004) fails to match the average shape of term structures of bond and equity risk premia. In addition, the term structure of Sharpe ratios of dividend strips is upward sloping under this calibration.

$$SR_n = \frac{(q - \frac{1}{\psi})\left(\frac{1 - \rho_x^n}{1 - \rho_x}\right)\frac{k_{c,2}(\gamma - \frac{1}{\psi})\sigma_x^2}{1 - k_{c,2}\rho_x}}{\sqrt{\left(\frac{1 - \rho_x^n}{1 - \rho_x}\right)^2\left(q - \frac{1}{\psi}\right)^2\sigma_x^2 + \sigma_d^2}}. \quad (48)$$

By dividing both the numerator and the denominator by  $1 - \rho_x^n$ , we can show that  $SR_n$  is an increasing function of  $n$ , which means the upward-sloping term structure of Sharpe ratios that are at odds with the data.

There are a couple of papers to address the term structure of equity risk premia without deviating from the standard calibration of preference parameters. Belo et al. (2015) introduce a time-varying procyclical leverage ratio  $q_t$  that positively co-varies with a shock to expected consumption growth. Since the leverage reverts to the mean in the long-run, dividend process is much riskier in the short-run than the model with the constant leverage ratio.<sup>12</sup> While this modification is shown to generate a downward-sloping term structure of equity risk premia even though investors still prefer the early resolution of uncertainty, it does not change the model's implications for the term structure of real bond risk premia. Marfè (2016) further develops this idea by assuming implicit wage insurance between firms and workers. In this model, stockholders provide insurance to wage earners in the short-run although both wages and profits co-move along the same trend in the long run. As a result, dividend is more volatile in the short-run than in the long-run. Therefore, the short-run dividend strip commands a higher risk premium than a long-run dividend strip. In addition, Marfè (2016) assumes that only

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<sup>12</sup>In fact, if  $q_t$  is a simple linear function of  $x_t$ , the model implies that  $z_{1,t}^d$  is a quadratic function of  $x_t$  and can be nested as a special case of our quadratic model.

stockholders participate in asset markets. Given their high exposures to the short-run dividend risk, stockholders are willing to pay extra money to purchase insurance against the short-run cash flow risk, driving down the yield of the short-run real bond. Hence, the term structure of real bond risk premia is upward-sloping on average in his model.

However, even these new models still cannot match the downward-sloping term structure of Sharpe ratios of dividend strips highlighted by van Binsbergen et al. (2012b). Because they generate higher equity risk premia in the short-run by amplifying cash flow risk but do not change the market price of risk, these models cannot generate the downward-sloping term structure of Sharpe ratios of dividend strips. Croce et al. (2015) address this issue by assuming investors' bounded rationality and imperfect information about the long-run shock ( $x_t$ ). In their model, investors cannot fully distinguish the short-run cash flow shock ( $\epsilon_{c,t+1}, \epsilon_{d,t+1}$ ) from the long-run shock  $x_t$ . They use a simple statistical model and try to filter out  $x_t$  from innovations to cash flows each period. Because they do not fully separate *iid* shocks from persistent shocks in the short-run although they can learn about them in the long-run, investors become more sensitive to the short-run cash flow risk, driving up the market price for risk of the short-run dividend strip. So this model generates the downward-sloping term structure of equity risk premia and Sharpe ratios of dividend strips. Our model can also generate this feature because  $x_t$  shock not only affects expected cash flow but also the market price of cash flow risk too.

While Croce et al. (2015) generate results broadly consistent with many stylized facts on the term structure of equity risk premia, they do not look into the term structure of bond risk premia. Recent studies show that long-run risks models can generate results consistent with stylized facts on bond market risk premia using the exposure of inflation shock to consumption risk (Bansal and Shaliastovich (2013), Creal and Wu (2015), Doh (2013), Song (2014) among others). When high inflation in the future is related to a negative shock to expected consumption growth, long-term nominal bonds whose real payoff decrease in case of high inflation are less valuable to investors. Therefore, they command higher risk premia. By allowing a non-zero correlation between two risk factors, our model can incorporate this negative correlation between expected inflation and expected consumption growth that contributes to the upward-sloping term structure of nominal bond risk premia.

Our quadratic model can overcome the shortcomings of the standard calibration of the simple long-run risks model in matching the term structure of different asset markets

by accommodating various extensions of the simple model in one general framework of the time-varying market price of risk. In the quadratic model, the mean excess return on an  $n$ -period real bond is given by

$$rpr_n^q = (\theta - 1)k_{c,2}(\lambda'_{c,1}\Omega_x B_{n-1} + 2 \times \text{trace}(\bar{H}_c\Omega_x\bar{C}_{n-1}E(X_tX'_t))). \quad (49)$$

Unlike the standard long-run risks model, we can set  $\lambda_{c,1}$  and  $\bar{H}_c$  independently of preference parameters such as  $\gamma$  and  $\psi$ . This flexibility allows us to match the term structure of real bond risk premia better than the standard approach. Similarly, our model implies mean excess return on an  $n$ -period ahead dividend strip and Sharpe ratio as follows:

$$\begin{aligned} rpd_n^q &= (1 - \theta)k_{c,2}(\lambda'_{c,1}\Omega_x B_{d,n-1} + 2 \times \text{trace}(\bar{H}_c\Omega_x\bar{C}_{d,n-1}E(X_tX'_t))) + \gamma\rho_{cd}\sigma_c^2, (50) \\ SR_n &= \frac{(1 - \theta)k_{c,2}(\lambda'_{c,1}\Omega_x B_{d,n-1} + 2 \times \text{trace}(\bar{H}_c\Omega_x\bar{C}_{d,n-1}E(X_tX'_t))) + \gamma\rho_{cd}\sigma_c^2}{\sqrt{B'_{d,n-1}\Omega_x B_{d,n-1} + \text{trace}(\bar{C}_{d,n-1}\Omega_x\bar{C}_{d,n-1}E(X_tX'_t)) + \sigma_d^2}} (51) \end{aligned}$$

By choosing  $\lambda_{d,1}$  and  $\bar{H}_d$  that determine  $B_{d,n}, \bar{C}_{d,n-1}$  independently of preference parameters  $\gamma$  and  $\psi$ , we can match the term structure of equity risk premia better than the standard approach. In the next section, we calibrate model parameters to illustrate these points quantitatively.

## 4 Quantitative Analysis

### 4.1 Data

To calibrate model parameters, we use U.S. data for cash flows, inflation, and financial variables. We compute moments of bond yields with maturities of 1 month, 6 month, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years. One month yield is from van Binsbergen et al. (2012b) while other yields are from Treasury constant maturity yields published by the Federal Reserve Board. The sample period is from February 1996 to October 2009. The real term structure of interest rates data are from McCulloch (2009), who construct real yield curve based on prices of Treasury Inflation-Protected Securities (TIPS). We compute sample moments for six different maturities (1, 2, 3, 5, 7, 10 year) using data

from April 1998 to October 2009. For consumption growth and inflation, we compute monthly growth rates of real consumption on non-durable goods and services and the price index of these components from February 1996 to October 2009. Real dividend growth and dividend strip prices are from van Binsbergen et al. (2012b) for same time period.

## 4.2 Calibration

We calibrate  $\lambda_{c,0}$ ,  $\lambda_{d,0}$ , and  $\bar{r}$  to exactly match model-implied unconditional mean values of  $E(\Delta c_{t+1})$ ,  $E(\Delta d_{t+1})$ , and  $E(\pi_{t+1})$  with sample moments conditional on all the other parameters. Parameters listed in Table 1 are set to match model-implied unconditional moments calculated by simulation with sample moments from the U.S. data.<sup>13</sup> Tables 2 and 3 illustrate target moments used in the calibration. In spite of the fact that we target far more moments than the number of parameters, the calibrated model is able to match sample moments fairly well.<sup>14</sup> The largest discrepancy is that the model implies a low persistence for the inflation process while the data indicate a moderately persistent process. In contrast, the dividend growth process is not persistent in the data but the model implies a moderately persistent process. Other than these two moments, volatilities and cross-correlations of consumption growth, dividend growth, and inflation are relatively well matched.

In addition, the model fits the average shape of the real and nominal term structure of interest rates and the term structure of dividend strip prices fairly well. As explained in the previous section, the standard calibration of long-run risks models tends to generate a downward-sloping term structure of real bond risk premia and an upward-sloping term structure of equity risk premia. Figures 2 and 3 show that our model can generate an upward-sloping term structure of bond risk premia together with a downward-sloping portion in the term structure of equity risk premia although our calibration of preference parameters still suggests that investors prefer the early resolution of uncertainty.<sup>15</sup> Assuming the log of price/cash flow ratios as quadratic functions of state variables is critical for this result because the term structure of risk premia would be at odds with

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<sup>13</sup>We generate 100,000 observation from the model using calibrated parameter values and compute model-implied moments from these simulated observations.

<sup>14</sup>We target 43 moments with 25 parameters.

<sup>15</sup>Because the infinite sum of dividend strip prices divided by the current dividend level must converge to a finite value, dividend strip prices must eventually decrease as the maturity increases. This implies that the term structure of equity risk premia should eventually upward-sloping although it can be downward sloping within a certain finite horizon.

the data in the linear model as discussed in the previous section. Figures 4 ~ 5 showing the term structure of bond and equity risk premia when we set all the quadratic coefficients ( $H_c, H_d$ ) to zeros confirm this analysis.

Our calibration of parameters governing the dynamics of state variables is also comparable to the standard calibration of long-run risks models, in that there are small but persistent components in cash flows and inflation. Because both eigenvalues of  $I + \Phi$  are greater than 0.92,  $X_t$  follows a fairly persistent process but  $\Omega_x$  is pretty small compared to  $\sigma_c^2$ ,  $\sigma_d^2$ , and  $\sigma_p^2$ . Therefore, the persistence of  $\Delta c_{t+1}$ ,  $\Delta d_{t+1}$ , and  $\pi_{t+1}$  are much smaller than those of  $X_t$ , implying that their variations are mostly driven by transitory components uncorrelated with  $X_t$ .

Our calibrated model generates decent results even for non-targeted moments. Especially, the Sharpe ratio of buying and holding long-run dividend strips or the aggregate market portfolio is much lower than that of buying and holding near-term dividend strips (van Binsbergen et al. (2012b)). We consider three different investment strategy.<sup>16</sup> The first one is to buy and hold a claim for dividends between 1 month and 24 months from now for six months. Investors can sell this claim six months from now while taking dividend streams maturing in between. The six month holding period return of this strategy in excess of the risk-free rate is denoted by  $exr_1$ . So this strategy focuses near-term dividend flows. The second strategy is to buy and hold a claim for dividends between 19 and 24 months from now for 6 months. The six month holding period return of this strategy in excess of the risk-free rate is denoted by  $exr_2$ . Finally, one can compute a similar 6 month holding period excess return for the aggregate market portfolio. Let's denote this excess return by  $exr_3$ . All these different returns can be formally expressed as follows:

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<sup>16</sup>van Binsbergen et al. (2012b) assume that dividends distributed are reinvested into dividend strips while we assume that dividends distributed are held as cash. However, the qualitative pattern of the term structure of excess returns (downward-sloping term structure of excess returns and Sharpe ratios) is similar, which is the focus of this exercise.

$$exr_1 = \frac{\sum_{j=1}^{18} P_{j,t+6}^d + \sum_{k=1}^6 D_{t+k}}{\sum_{j=1}^{24} P_{j,t}^d}, \quad (52)$$

$$exr_2 = \frac{\sum_{j=13}^{18} P_{j,t+6}^d}{\sum_{j=19}^{24} P_{j,t}^d}, \quad (53)$$

$$exr_3 = \frac{\sum_{j=1}^{\infty} P_{j,t+6}^d + \sum_{k=1}^6 D_{t+k}}{\sum_{j=1}^{\infty} P_{j,t}^d}. \quad (54)$$

Table 4 provides information related to moments of excess returns of these different strategies. The data suggest that not only the level of  $exr_2$  or  $exr_3$  is lower than that of  $exr_1$ , but also the Sharpe ratio of the first strategy is higher than those of others, implying the downward-sloping term structure of Sharpe ratio. As discussed in the previous section, the standard calibration of the long-run risks model has difficulty in matching this pattern. While we do not directly target the term structure of Sharpe ratio, the calibrated model generates a similar pattern. On the other hand, once we shut down time-varying market price of risk by setting  $H_c$  and  $H_d$  to zeros, we do not replicate this pattern.

To sum it up, the calibrated version of our quadratic asset pricing model generates results consistent with a variety of stylized facts found in the term structure of bond and equity risk premia even with the standard calibration of preference parameters in the context of long-run risks models. As discussed in the previous section, the existing literature tries to reconcile the model's implications for the term structure of one class of assets with the data. Our paper shows that incorporating time-varying market prices of risks to an otherwise standard long-run risks model goes a long way to reconcile the model's implications with many stylized facts on term structure of bond and equity risk premia that have been found to be challenging before.

### 4.3 Economic interpretation of state variables

A prominent shortcoming in our framework is that the economic interpretation of state variables may not be straightforward. Although our model can interpret  $x_{1,t}$  as expected consumption growth and  $x_{2,t}$  as expected inflation with certain restrictions on parameters, the calibrated parameter values do not satisfy these restrictions. We pursue the economic interpretation of each component in  $X_t$  by looking at impulse-responses

of cash flows and inflation to a positive one standard deviation shock to  $X_t$ . Because our model has nonlinearities with respect to  $X_t$ , we compute the generalized impulse response functions following Koop et al. (1996) and describe several quantile statistics related to responses in Figure 6. A shock to  $x_{1,t}$  generates a positive co-movement of consumption growth and inflation. The finding suggests that it can be interpreted as a kind of the aggregate demand shock in terms of its impact on macro variables. On the other hand, a shock to  $x_{2,t}$  leads to the opposite responses of consumption growth and inflation. Inflation declines in response to a positive shock to  $x_{2,t}$  but consumption growth increases. The magnitude of responses suggest that  $x_{2,t}$  is likely to be a more dominant risk factor in explaining cash flows and inflation.

The responses of consumption growth and inflation to a shock to  $x_{2,t}$  seem to be puzzling at first because it is more likely to be a nominal shock attached to monetary policy rule under our specification. However, the clear distinction between real and nominal risk factors is more nuanced under our calibration because we have non-zero parameters governing the interaction between  $x_{1,t}$  and  $x_{2,t}$  such as  $\Omega_x^{1,2}$  and  $\phi_{12}$ . The calibrated value for  $\phi_{12}$  is negative, meaning that a positive shock to  $x_{2,t}$  negatively affects  $x_{1,t}$  in the future. Given  $\lambda_{c,2} < 0$ , a positive shock to  $x_{2,t}$  decreases the current value of wealth-consumption ratio so that the wealth-consumption ratio is expected to increase in the future as the impact of a positive shock dies out. When  $\phi_{12}$  is zero and investors prefer the early resolution of uncertainty with the intertemporal elasticity greater than 1 ( $\gamma > 1 > \frac{1}{\psi}$ ), this channel alone negatively affects  $E_t(e^{m_{t+1}+r_{c,t+1}})$ . Because the no-arbitrage condition for  $E_t(e^{m_{t+1}+r_{c,t+1}}) = 1$  must hold for any value of  $x_{2,t}$ , the decrease in  $m_{t+1}+r_{c,t+1}$  should be offset by the decreased expected consumption growth which pushes up  $m_{t+1}+r_{c,t+1}$  by  $(1-\gamma)g_c(X_t)$ . However, when  $\phi_{12}$  is negative and  $\lambda_{d,1}$  is positive, there is another channel that pushes up  $m_{t+1}+r_{c,t+1}$ . In our calibration, the second channel is dominant and for this reason, expected consumption growth must go up to decrease  $m_{t+1}+r_{c,t+1}$  in response to a positive shock to  $x_{2,t}$  if the no-arbitrage condition should hold. Otherwise, a positive shock to  $x_{2,t}$  will drive down the current price of consumption claim so much that the asset cannot be traded in equilibrium.

Interestingly, responses of cash flows and inflation to  $X_t$  change quantitatively but not qualitatively even if we shut down quadratic coefficients in price/cash flow ratios as shown in Figure 7.<sup>17</sup> Hence, the time-varying market prices of risks have first-order

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<sup>17</sup>The response of dividend growth to a positive shock to  $x_{1,t}$  changes the sign but the quantitative magnitude involved is so small that it does not seem to be a substantial change.

impacts on the dynamics of risk premia but only second-order impacts on the dynamics of cash flows and inflation.

## 5 Conclusion

Jointly explaining the term structure of equity and interest rates without hampering macroeconomic implications in the context of an equilibrium asset pricing model has been challenging. We develop an equilibrium asset pricing model with recursive preferences and calibrate the model to match major stylized facts of the term structure of equity and interest rates as well as moments for cash flows and inflation. The linear approximation of the log of wealth/consumption ratio in the pricing kernel implied by recursive preferences fails to generate the downward sloping term structure of equity risk premia and the upward-sloping term structure of real bond risk premia at the same time under a reasonable calibration of preference parameters. However, when we use the quadratic approximation of the log of wealth/consumption ratio, we can obtain the additional flexibility due to the time-varying market price of risk, which enables us to fit various stylized facts of term structure of risk premia with plausible values for preference parameters. The calibrated model can replicate the pattern that the Sharpe ratio of the aggregate stock market excess return is lower than that of the investment strategy focusing on the near-term strips although we do not directly target such a pattern in the calibration. Given the fact that the standard long-run risks model has difficulty in matching this pattern even with various extensions, our results are quite illuminating.

The impulse-response analysis of the model suggests that a shock generating a negative co-movement of consumption growth and inflation can explain bulk of the variations in cash flows and inflation while a shock generating a positive co-movement of consumption growth and inflation plays a minor role. This finding suggests that factors behaving like aggregate supply shocks explain most of time variations in cash flows and inflation as well as the term structure of risk premia.

Interestingly, the time-varying market price of risk changes the model's implications for asset prices both quantitatively and qualitatively but does not affect the model's implications for the dynamics of cash flows and inflation materially. This finding is consistent with the commonly held view that the presence of nonlinearities would be more important for asset price movements than movements in cash flows and inflation.

Our model can be extended in various dimensions to improve the model's implica-

tions for asset price and macro variables. First of all, our parsimonious structure rules out time-varying volatilities in state variables themselves. This assumption may be at odds with the literature emphasizing macro variance risk factors in explaining facts on the term structure of risk premia (Bekaert et al. (2016), for example). In addition, volatility shocks can alleviate a too strong predictability of dividend growth. Second, restrictions on market price of risks that would make the economic interpretation of  $X_t$  more transparent may be useful for understanding sources of fluctuations in risk premia. Finally, the calibration focuses on matching unconditional moments of the term structure of risk premia. However, Ait-Sahalia et al. (2015) notice that the slope of the term structure of risk premia shifts over business cycles. Extending our model to match conditional moments regarding the term structure of risk premia seems to be a challenging but worthwhile task.

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## Appendix

### A.1: Incorporating time-varying market price of risk in the standard approach

Under the standard approach, we first specify expected consumption growth as a function of state variables and determine the coefficients in the wealth-consumption ratio using Euler equation for the return on consumption claim. Consider the following one-factor long-run risks model in which expected consumption growth is given by a quadratic function of the state variable  $x_t$ .

$$\begin{aligned}
 g_c(x_t) &= g_{c,0} + g_{c,1}x_t + \frac{1}{2}W_c x_t^2, \\
 f_c(x_t) &= \lambda_{c,0} + \lambda_{c,1}x_t + \frac{1}{2}H_c x_t^2, \\
 r_{c,t+1} &= (k_{c,0} - k_{c,1}\lambda_{c,0} + g_{c,0}) + (-k_{c,2}\lambda_{c,1} + k_{c,2}\lambda_{c,1}\Phi + g_{c,1})x_t \\
 &\quad + \frac{1}{2}(k_{c,1}H_c + 2k_{c,2}H_c\Phi + W_c)x_t^2 + k_{c,2}(\lambda_{c,1} + H_c x_t)\epsilon_{x,t+1} + \epsilon_{c,t+1}, \\
 m_{t+1} &= \theta \log \delta + (\theta - 1)r_{c,t+1} - \frac{\theta}{\psi}(g_c(x_t) + \epsilon_{c,t+1}).
 \end{aligned} \tag{55}$$

Using the Euler equations for  $r_{c,t+1}$ , which is  $E_t(e^{m_{t+1}+r_{c,t+1}}) = 1$ , we can compute  $\lambda_{c,0}$ ,  $\lambda_{c,1}$  and  $H_c$  as functions of other model parameters. In this case,  $H_c$  is given as the solution to the following quadratic equation.

$$\theta^2 k_{c,2}^2 \sigma_x^2 H_c^2 + \theta(2k_{c,2}\Phi - k_{c,1})H_c + (1 - \gamma)W_c = 0. \tag{56}$$

In a general multi-factor model, the expression becomes more complicated and the analytical solution is not available.  $H_c$  must be found as a numerical solution of the matrix quadratic equation.

In contrast, if we start from specifying  $f_c(x_t)$  as a quadratic function of  $x_t$ , we can derive  $g_{c,0}$ ,  $g_{c,1}$  and  $W_c$  easily as we have shown in the main context. Our approach is analytically tractable even for a multiple risk factor case. Therefore, our approach would be preferable in terms of tractability if we want to introduce affine market prices of risks by assuming the wealth-consumption ratio as a quadratic function of state variables.

## A.2: Convergence conditions for coefficients of bond and dividend strip prices

To ensure that the long end of bond yield curves and dividend strips is well defined, we check if coefficients in asset prices converge as the maturity increases at calibrated parameter values. First, we find solutions for  $\lim_{n \rightarrow \infty} C_{\$,n}$ ,  $\lim_{n \rightarrow \infty} C_n$ ,  $\lim_{n \rightarrow \infty} C_{d,n}$  by solving matrix quadratic equations. Second, we check eigenvalues of  $(I + \Phi - \Omega_x(\bar{C}_{\$, \infty} - (\theta - 1)k_{c,2}\bar{H}_c))$ ,  $(I + \Phi - \Omega_x(\bar{C}_{\infty} - (\theta - 1)k_{c,2}\bar{H}_c))$ , and  $(I + \Phi + \Omega_x(\bar{C}_{d, \infty} + (\theta - 1)k_{c,2}\bar{H}_c))$ . Finally, for dividend strips, we need to make sure the aggregate price-dividend ratio is also well defined. The necessary and sufficient condition for this is that  $P_{n,t}^d$  goes to zero as  $n$  goes to  $\infty$ . This can be satisfied if  $a_{d,n} - a_{d,n-1}$  becomes negative as  $n$  goes to  $\infty$ . This condition holds at our calibrated parameter values because  $\lambda_{d,0} + \frac{(B_{d,\infty} + 2(\theta - 1)k_{c,2}\lambda_{c,1})'\Omega_x B_{d,\infty}}{2} < 0$ .

Table 1: PARAMETER CALIBRATION

Parameter	Value
$\lambda_{c,11}$	9.8983
$\lambda_{c,12}$	-14.524
$\lambda_{d,11}$	-0.1008
$\lambda_{d,12}$	-10.8342
$h_{c,11}$	-18.887
$h_{c,22}$	939.6492
$h_{d,11}$	-12.2408
$h_{d,22}$	12562
$\gamma$	49.9256
$\psi$	1.4062
$\delta$	0.9997
$\alpha_c$	0.00035
$\alpha_\pi$	1.9882
$\tilde{\phi}_{11}$	-0.001
$\tilde{\phi}_{12}$	-0.1303
$\tilde{\phi}_{22}$	-0.0613
$\sigma_{x,1}$	$1.2 \times 10^{-6}$
$\sigma_{x,2}$	$2.72 \times 10^{-4}$
$\sigma_c$	0.0024
$\sigma_d$	0.0068
$\sigma_\pi$	0.0022
$\Omega_x^{1,2}$	$-1.46 \times 10^{-13}$
$\rho_{cd}$	1.9782

Table 2: TARGET MOMENTS: CASH FLOWS AND INFLATION

Moment	Data	Model
E ( $\Delta c_{t+1}$ )	2.520	2.520
E ( $\Delta d_{t+1}$ )	1.188	1.188
E ( $\pi_{t+1}$ )	2.520	2.520
Std ( $\Delta c_{t+1}$ )	3.083	3.125
Std ( $\Delta d_{t+1}$ )	15.25	15.294
Std ( $\pi_{t+1}$ )	2.84	2.81
AR ( $\Delta c_{t+1}$ )	-0.039	0.174
AR ( $\Delta d_{t+1}$ )	0.048	0.535
AR ( $\pi_{t+1}$ )	0.444	0.153
Corr ( $\Delta c_{t+1}, \Delta d_{t+1}$ )	0.124	0.238
Corr ( $\Delta c_{t+1}, \pi_{t+1}$ )	0.017	0.091
Corr ( $\Delta d_{t+1}, \pi_{t+1}$ )	-0.138	0.067

Table 3: TARGET MOMENTS: FINANCIAL VARIABLES

Moment	Data	Model	Moment	Data	Model
E ( $i_{1,t}$ )	3.36	3.43	E( $i_{6,t}$ )	3.60	3.50
E ( $i_{12,t}$ )	3.72	3.80	E( $i_{24,t}$ )	3.96	4.09
E ( $i_{36,t}$ )	4.2	4.25	E( $i_{60,t}$ )	4.44	4.40
E ( $i_{84,t}$ )	4.68	4.53	E( $i_{120,t}$ )	4.92	4.89
Std ( $i_{1,t}$ )	1.871	2.044	Std ( $i_{6,t}$ )	1.889	1.959
Std ( $i_{12,t}$ )	1.822	1.859	Std( $i_{24,t}$ )	1.725	1.665
Std ( $i_{36,t}$ )	1.581	1.486	Std( $i_{60,t}$ )	1.323	1.186
Std ( $i_{84,t}$ )	1.182	1.003	Std( $i_{120,t}$ )	1.002	1.025
E ( $r_{12,t}$ )	1.704	1.376	E( $r_{24,t}$ )	1.872	1.827
E ( $r_{36,t}$ )	2.028	2.092	E( $r_{60,t}$ )	2.328	2.365
E ( $r_{84,t}$ )	2.508	2.526	E( $r_{120,t}$ )	2.628	2.778
E ( $\sum_{j=1}^6 e^{z_{j,t}^d}$ )	6.146	5.927	E ( $\sum_{j=1}^1 2e^{z_{j,t}^d}$ )	11.986	11.603
E ( $\sum_{j=1}^{18} e^{z_{j,t}^d}$ )	17.651	17.079	E ( $\sum_{j=1}^2 4e^{z_{j,t}^d}$ )	23.356	22.488
E ( $\sum_{j=1}^{600} e^{z_{j,t}^d}$ )	725.66	683.44	E ( $f_c(X_t)$ )	6.8988	6.8214
E ( $exr_1$ )	5.909	5.755	E ( $exr_2$ )	3.367	3.429
E( $exr_3$ )	3.009	2.989			

Notes: Data for  $\sum_{j=1}^{600} e^{z_{j,t}^d}$  is the price of S \$ P 500 hunded index. E ( $f_c(X_t)$ ) from Lustig et al. (2013).

Table 4: TERM STRUCTURE OF EXCESS RETURNS ON DIVIDEND STRIPS

Variable	<i>exr</i> <sub>1</sub>	<i>exr</i> <sub>2</sub>	<i>exr</i> <sub>3</sub>
Mean Excess Return (Data)	5.909	3.367	3.009
Mean Excess Return (Model)	5.755	3.429	2.989
Mean Excess Return (Model: $H_c = H_d = 0$ )	16.366	30.31	28.384
Standard Deviation of Excess Return (Data)	29.211	41.614	26.198
Standard Deviation of Excess Return (Model)	11.92	16.669	17.478
Standard Deviation of Excess Return (Model: $H_c = H_d = 0$ )	9.609	17.513	16.624
Sharpe Ratio (Data)	0.202	0.081	0.115
Sharpe Ratio (Model)	0.483	0.206	0.171
Sharpe Ratio (Model: $H_c = H_d = 0$ )	1.703	1.731	1.707

Figure 1: MEAN BOND YIELD CURVE

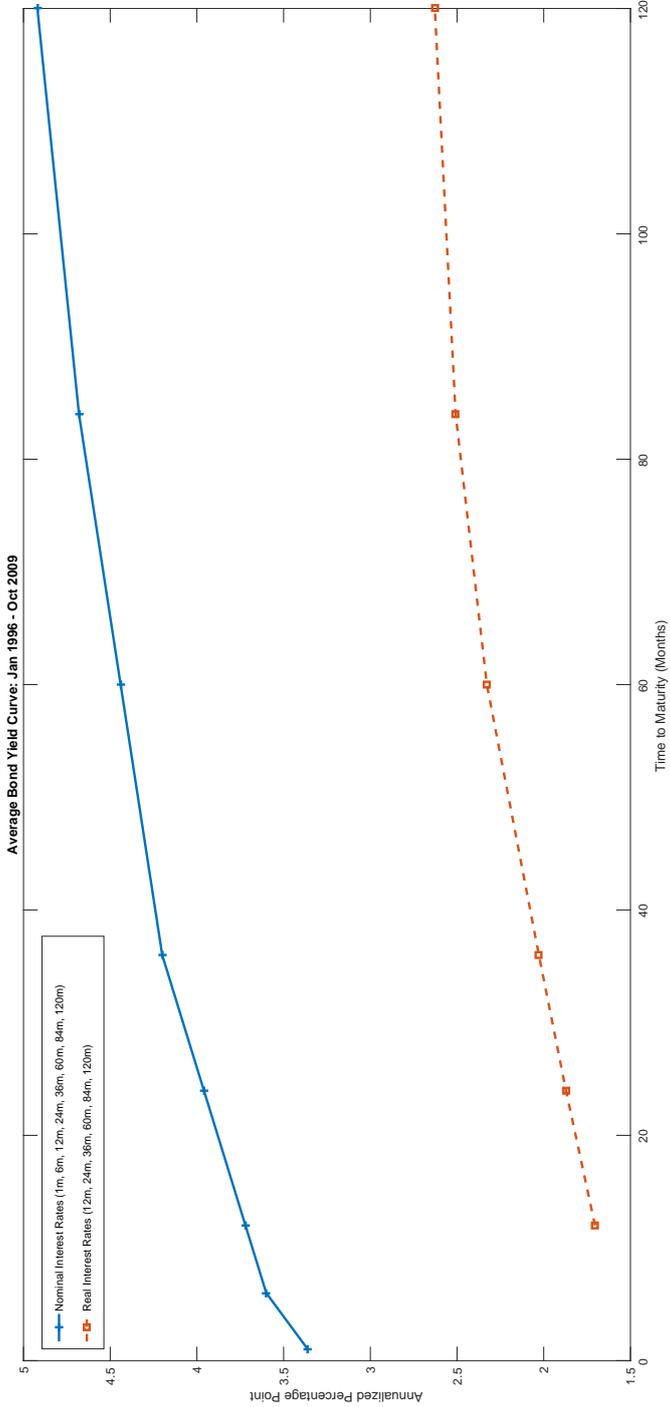


Figure 2: TERM STRUCTURE OF BOND RISK PREMIA

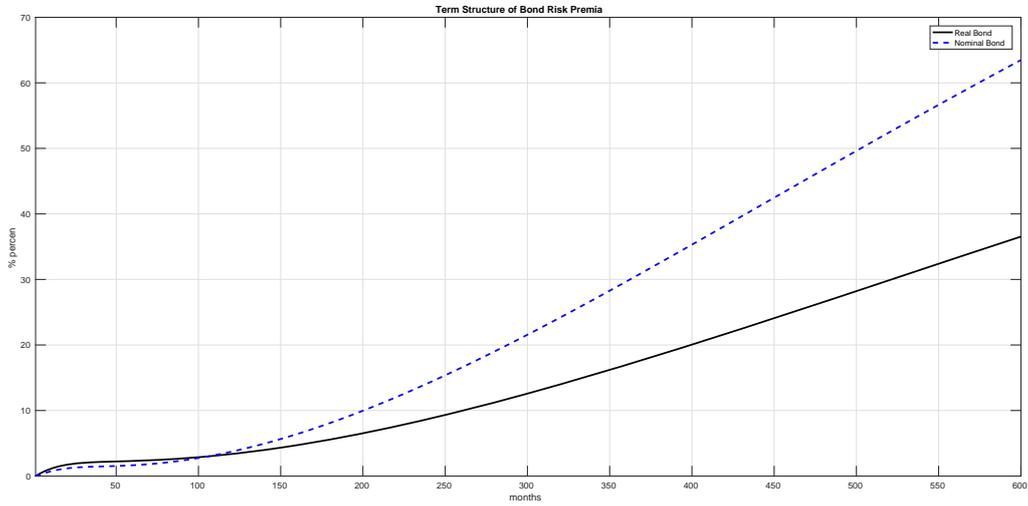


Figure 3: TERM STRUCTURE OF EQUITY RISK PREMIA

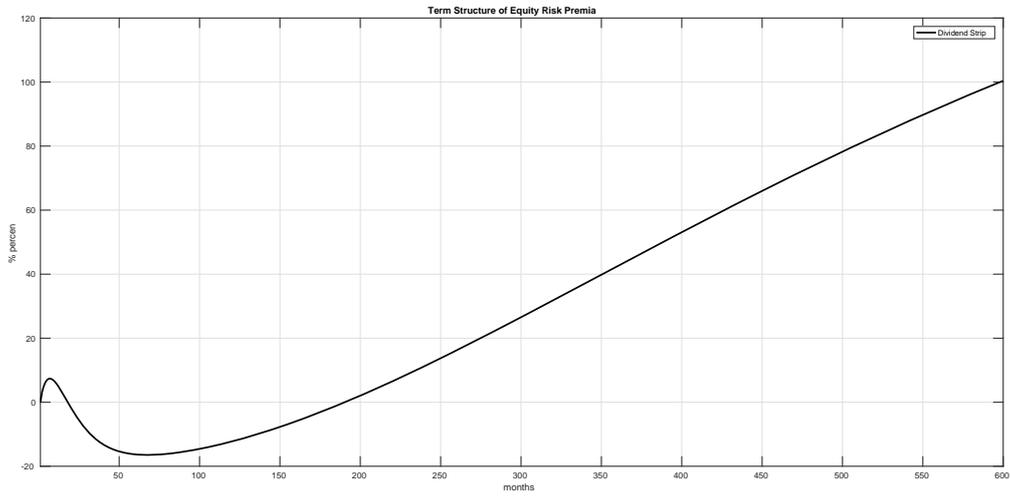


Figure 4: TERM STRUCTURE OF BOND RISK PREMIA:  $H_c = H_d = 0$

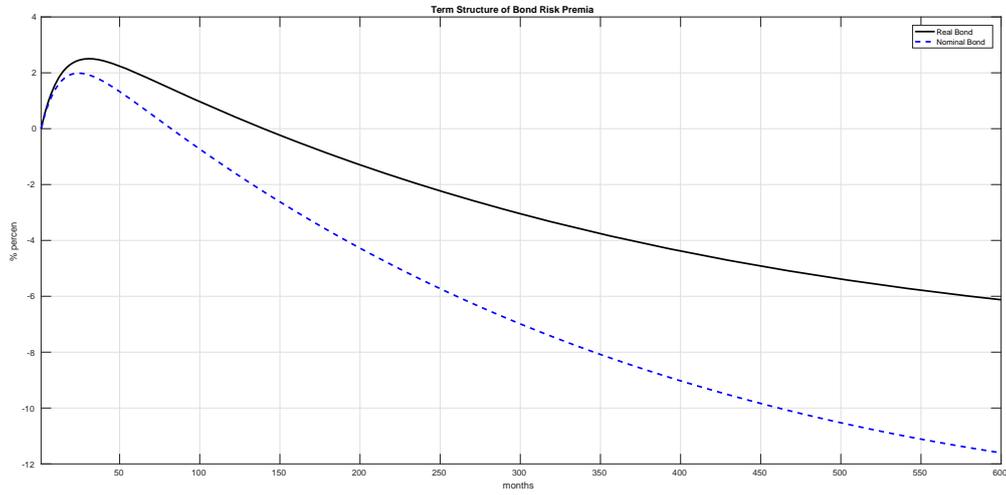


Figure 5: TERM STRUCTURE OF EQUITY RISK PREMIA:  $H_c = H_d = 0$

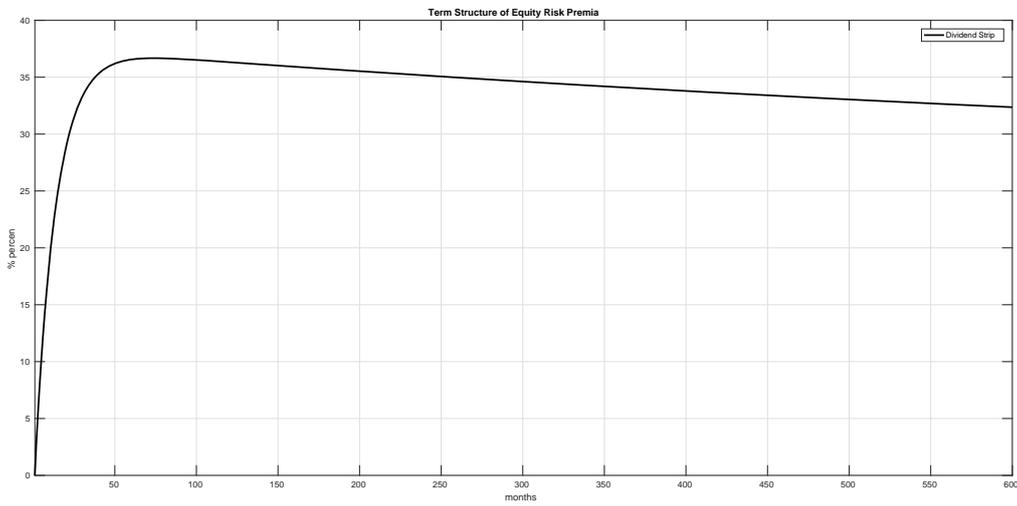


Figure 6: IMPULSE RESPONSES OF  $(\Delta c_{t+1}, \Delta d_{t+1}, \pi_{t+1})$  TO  $X_t$

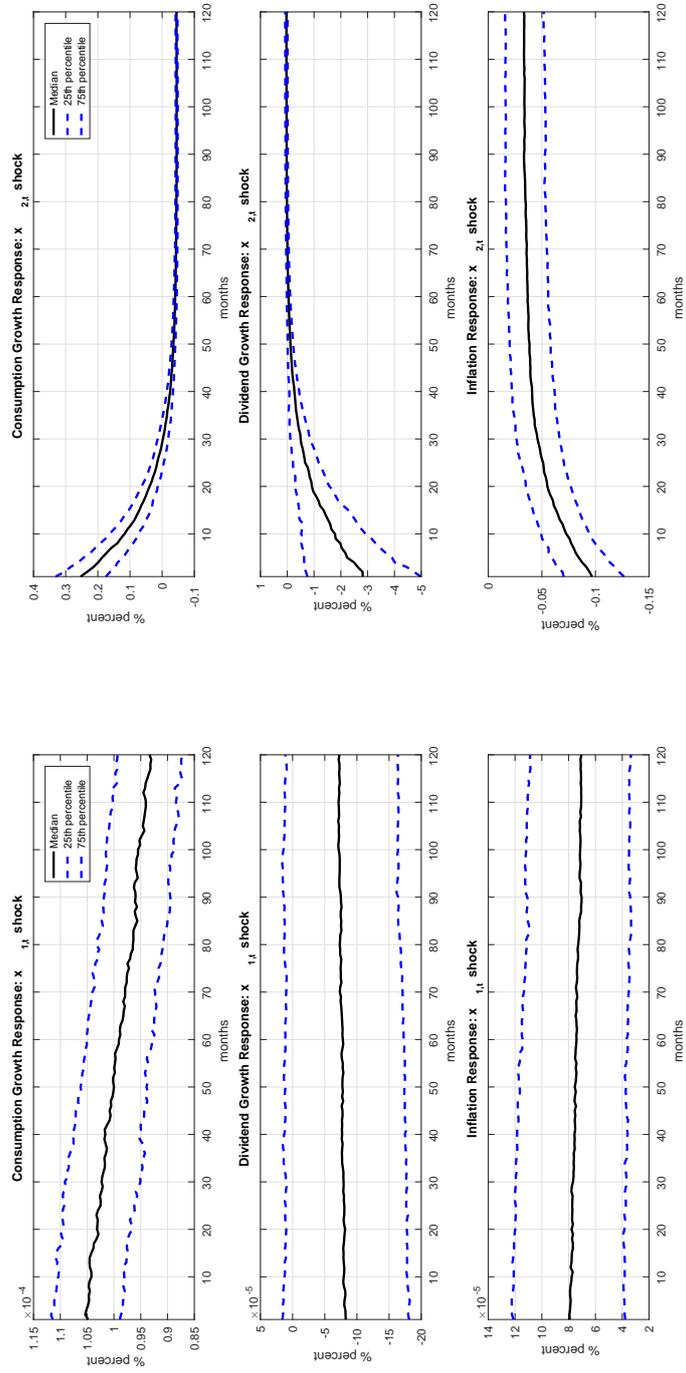


Figure 7: IMPULSE RESPONSES OF  $(\Delta c_{t+1}, \Delta d_{t+1}, \pi_{t+1})$  TO  $X_t$ :  $H_c = H_d = 0$

