THE CONDUCT OF MONETARY POLICY WITH A SHRINKING STOCK OF GOVERNMENT DEBT

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Abstract

In many countries, government-budget surpluses have led to a decline in the amount of federal government debt outstanding. This paper considers the consequences of this development for a central bank that conducts monetary policy through open market operations in treasury debt. A model is presented in which a treasury taxes, spends, and issues debt; a central bank conducts monetary policy through open market operations; and banks are intermediaries for all private savings. The model suggests potentially severe consequences from a shrinking stock of government debt in the absence of a change in the conduct of monetary policy. Specifically, the nominal interest rate and the inflation rate cannot be below their seigniorage-maximizing levels. In effect, a small stock of debt combined with restrictions on a central bank's portfolio can put the economy on the Pareto inferior side of the seigniorage Laffer curve, with an unnecessarily high inflation rate and nominal interest rate. Moreover, if the government also runs a primary budget deficit, equilibrium can fail to exist. The model presented can yield estimates of how much debt must be outstanding to avoid each situation. Discount-window lending is a feasible—and desirable—alternative method for conducting monetary policy. It relaxes any restrictions on the attainable set of interest rates and inflation rates implied by a decline in the stock of government debt outstanding. Unless the economy is on the Pareto inferior side of the Laffer curve, welfare is higher when discount-window loans are made at market-determined interest rates.

Keywords: Monetary policy, Fiscal policy, Government debt, Discount Window

JEL Codes: E4, E5, E6, H6

1. Introduction

In many countries, government budget surpluses have led to a decline in the amount of federal government debt outstanding (Kopcke 2001). This trend has been especially dramatic—and unexpected—in the United States in recent years. U.S. federal government debt in the hands of the public, including the Federal Reserve System, was about $3.4 trillion for fiscal year 2000, down from its peak of almost $3.8 trillion in fiscal year 1997. This downward trend is expected to continue. The Congressional Budget Office (2001, Table 1-1) projects federal government debt in the hands of the public to shrink to $2.5 trillion by 2003, to $1.7 trillion by 2005, and to less than $0.9 trillion by 2010.

Such enormous reductions in the stock of outstanding debt could pose significant problems for a central bank, depending on how it conducts monetary policy. In fact, the Federal Reserve System has considered the possibility that it might face some special challenges, given its current operating procedures (Minutes of the FOMC, 2001). The Fed currently conducts monetary policy almost exclusively through open market operations in federal government debt. In addition, it limits its holdings of government debt of various maturities. As recently as 1990, the Fed held as little as 10 percent of all federal debt in the hands of the public. That share is closer to 15 percent today and is projected to grow dramatically. Under the Fed's operating procedures, some projections suggest that the Fed will hit its self-imposed limit on government bond holdings by 2003 (e.g., projections by Goldman Sachs economists Dudley and Youngdahl, 2000).

In light of these prospects, this paper addresses the following questions. What implications does the reduction in the stock of government debt outstanding have for the conduct of monetary policy? In particular, what implications does it have for the feasibility of attaining various inflation rates or nominal interest rates? Is there any rationale for a central bank limiting the fraction of outstanding government debt that it holds in its portfolio, as the Federal Reserve System currently does? Can monetary policy objectives that are currently accomplished through open market operations instead be accomplished through discount-window lending? If so, how should the discount rate be set relative to market interest rates?

The vehicle for addressing these questions is a model in which the treasury taxes, spends, and issues debt, and the central bank conducts monetary policy either through open market operations alone, or through a combination of open market operations and discount-window lending. As in Townsend (1980, 1987), spatial separation and limited communication create a transactions role for currency, and as in Diamond and Dybvig (1983), banks arise endogenously to insure agents against the effects of random shocks to their demand for liquid assets. More specifically, agents are divided between two distinct locations, and the opportunities for communication between these locations are limited. In each period, agents consume and decide how to allocate their savings between money, government bonds, and physical capital (storage). All savings are intermediated through the banking system. After making their investment decisions, some agents will find themselves randomly relocated from one location to another. Agents who are not relocated remain in contact with their bank, and hence can avoid using currency to make transactions. In contrast, agents who are relocated lose contact with their bank, and thus require currency to make purchases. Moreover, since the event of being relocated requires agents to transact with cash instead of other, higher-yielding assets, agents will want to be insured against the risk of being relocated and, by implication, the risk of having to convert other assets into currency to conduct transactions. Banks provide this insurance by using some of the deposits they accept to acquire cash reserves that they can use to pay relocated agents. The rest of their funds they invest in the economy’s two other primary assets—government bonds and physical storage. In equilibrium, banks’ demand for cash reserves depends on the opportunity cost of holding reserves. If the nominal interest rate (and thus, in this model, the inflation rate) is low, banks will wish to hold relatively large stocks of real cash reserves.

This latter observation has a potentially strong implication for a central bank that must back the outstanding stock of base money with a shrinking stock of government debt. If the stock of government debt outstanding is too small, the central bank will not be able to satisfy the demand for reserves that prevails when nominal interest rates are low. As a result, there will be a strict lower bound on the equilibrium nominal interest rate and inflation rate. Indeed, this lower bound will be such that the economy is forced to "the wrong side"—the Pareto inferior side—of the seigniorage Laffer curve, with the inflation rate and nominal interest rate both unnecessarily high. In addition, the model indicates the possibility of even more severe consequences for economies with primary government budget deficits: an equilibrium might fail to exist if the stock of government debt outstanding is too low.

As would be expected, the problems arising from a shrinking stock of government debt are exacerbated if the central bank limits the fraction of debt outstanding that it can hold. Such a limitation restricts further the amount of liquidity that the central bank can provide. It therefore requires corresponding increases in the amount of debt outstanding for an equilibrium on the Pareto superior side of the Laffer curve to be feasible. This model thus provides no rationale for a central bank to limit its holdings of government debt.

The model can be used to estimate how much debt must be outstanding to prevent the existence of binding lower limits on the nominal interest rate. Under current Federal Reserve operating procedures and current projections of the time path of the stock of government debt, the model indicates that by 2005 the stock of debt will be sufficiently small to force the economy to the Pareto inferior side of the Laffer curve.

Fortunately, any inability of the central bank to provide liquidity through open market operations alone can be rectified if the central bank is willing to use discount-window lending as an instrument of monetary policy. The Federal Reserve, for example, conducted policy primarily by adjusting discount-window credit during its early years. The model shows that use of a discount window can exactly replicate the equilibria feasible through open market operations alone if the central bank sets the discount rate at the market interest rate. This assumes, of course, that banks are willing to use the discount window. In the U.S., discount-window use has been extremely low since the mid-1980s, so something might have to be done to increase discount-window activity for the window to be an effective policy tool.[1] One possibility is to subsidize discount-window borrowing further, beyond the current subsidy of about 50 basis points. The model shows, however, that subsidized discount-window borrowing can increase steady-state welfare only if the economy is on the Pareto inferior side of the Laffer curve. If, instead, the economy is on the Pareto superior side of the curve—which is presumably the desired result of supplementing open market operations with discount-window lending—then a subsidized discount rate necessarily reduces steady-state welfare. These findings suggest that if monetary policy is to be conducted through the discount window, then there is a strong case for the loans to be made at market rates of interest.

The remainder of the paper proceeds as follows. Section 2 describes the environment. Sections 3 and 4 consider the implications for equilibrium of a shrinking stock of government debt, first in the case where the treasury is active and the central bank passive in terms of which institution moves first in setting its policy variables (following Leeper 1991), and second in the case where the central bank is active. The alternative of conducting monetary policy through the extension of discount-window credit is the subject of Section 5. Section 6 concludes.

2. The Environment

2.1 Private Agents

Consider an infinite-horizon economy, with $t = 1,2...$ indexing time. The economy consists of two identical islands, each inhabited by an infinite sequence of two-period-lived overlapping generations. At the start of each date, each generation has a continuum of ex ante identical young agents of measure one. In addition, at $t = 1$ there is an initial old generation in each location.

There is a single consumption good available at each date. Agents are endowed with

[1] For 2000, total discount-window borrowing amounted to less than 0.2 percent of total Federal Reserve assets (Board of Governors, 2001, p. 320).

$\omega > 0$ units of this good when young and before taxes are imposed. They are endowed with nothing when old. For simplicity, young agents derive no utility from consumption. With c denoting the quantity of the good consumed when old, an agent's lifetime utility is

$$u(c) = \frac{c^{1-\rho}}{1-\rho}; \quad \rho \in (0,1).$$

Agents have access to a common linear storage technology that allows them to transfer wealth across periods. One unit invested at date t yields $x > 1$ units of consumption at $t + 1$.

At the beginning of each date, agents can communicate and trade only with other agents inhabiting the same location. This limited ability to communicate in conjunction with the spatial separation generates a transaction role for currency as follows. Transactions can involve any of the economy's three primary assets: storage (investment), government bonds, and currency. Thus, at the beginning of each date, young agents either directly or indirectly store goods and acquire government liabilities in the form of bonds or currency. Once these portfolio allocations occur, a fraction π of young agents in each location is selected at random to move to the other island. The value π is thus the probability of relocation and is common knowledge. However, which individuals will have to relocate is not known at the beginning of a period.

The primary assets have some features that make transacting difficult for young agents who must relocate. Specifically, stored goods cannot be transported between locations. This could be the case because the returns to storage have not been realized at the time relocation occurs and the investment process cannot be interrupted, or because the cost of transporting the goods is prohibitive. Likewise, government bonds either are nonnegotiable or, as is typically true in practice, they are issued only in denominations too large to be used in individual transactions. In either case, the implication is that agents cannot transact with government bonds. As a result, agents who are relocated require currency to make purchases in their new location. Thus, as in Townsend (1980, 1987), spatial separation and limited communication create a role for money in transactions.

The event of being relocated forces agents to liquidate claims to high-yield assets (bonds and investments) in exchange for low-yield currency. Relocation thus acts much as the liquiditypreference shock in the Diamond and Dybvig (1983) model, and agents will wish to be insured against being relocated. Banks that accept deposits and hold the economy's primary assets directly can efficiently provide this insurance (Greenwood and Smith 1997). Thus, agents will choose to do all their saving through such intermediaries.

At the beginning of each period, young agents will deposit their after-tax endowment with a bank. The bank then allocates its portfolio between currency, government bonds, and storage, and chooses the rates of return to pay agents as a function of whether those agents relocate. (The next section describes the bank's problem in greater detail.) After these decisions are made, the specific identities of the agents to be relocated are revealed. Relocated agents then contact their bank in a decentralized manner, exchanging their deposit claims for cash. The currency obtained is then carried to the new location, where it will be used to purchase consumption goods the following period. For agents who are not relocated—and thus who remain in contact with their bank—currency is not required to make purchases. These agents then become residual claimants on their bank's interest-earning assets. This timing of events is depicted in Figure 1 below.

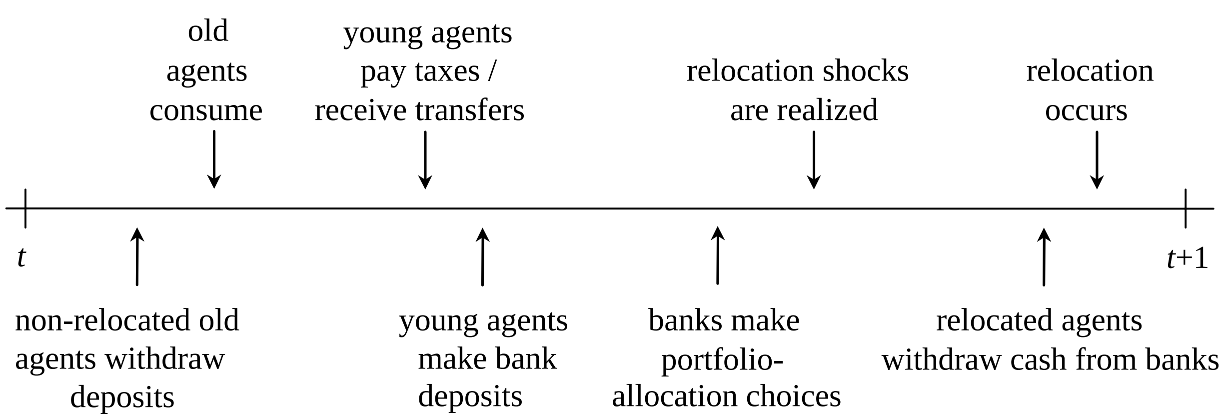

Figure 1 — The Timing of Economic Activity

2.2 The Government

In addition to the agents just described, the economy has an active government sector. This sector can be thought of as having two separate entities: the treasury and the central bank.

At each date, the treasury has an exogenously given level of real expenditures per capita, g_t, and levies a real lump-sum tax (transfer) on each young agent of $\tau_t > (<) 0$. Thus, its real primary budget deficit per capita is $g_t - \tau_t$ at date t. The treasury also issues government bonds with a nominal value of B_t at t. If p_t is the time t price level, the real value of treasury debt outstanding at the end of date t is B_t / p_t. Government bonds can be held either by the private sector or by the central bank. The real value of government bonds demanded by private agents and by the central bank at t is denoted b_t^p and b_t^c, respectively. The total stock of government debt demanded at t is $b_t \equiv b_t^p + b_t^c$.

The central bank issues fiat currency. At date t the per capita value of the monetary base outstanding is M_t in nominal terms and M_t / p_t in real terms. The central bank's balance-sheet constraint requires that the value of the central bank's outstanding liabilities not exceed the value of its holdings of government debt. That is,

$$\frac{M_t}{p_t} \le b_t^c, \quad t \ge 1. \tag{1}$$

In addition, the central bank's holdings of government bonds are limited by the stock of government bonds outstanding:

$$0 \le b_t^c \le \frac{B_t}{p_t}, \quad t \ge 1.$$

As in the United States, the central bank rebates to the treasury all interest earned on its holdings of government bonds after covering its expenses (assumed to be zero), but retains the principal. If R_t is the gross real rate of interest paid on government debt between t and $t + 1$ and T_t is the nominal value of the rebate at time t, then the real rebate is

$$\frac{T_t}{p_t} \equiv b_{t-1}^c \left(R_{t-1} - \frac{p_{t-1}}{p_t} \right). \tag{2}$$

Given the central bank's behavior, the treasury's budget constraint can be written as

$$g_t - \tau_t = \frac{B_t}{p_t} - R_{t-1}\frac{B_{t-1}}{p_{t-1}} + \frac{T_t}{p_t}. \quad (3)$$

Using (2) in (3) yields

$$g_t - \tau_t = \frac{B_t}{p_t} - R_{t-1}\left(\frac{B_{t-1}}{p_{t-1}} - b_{t-1}^c\right) - \frac{p_{t-1}}{p_t}b_{t-1}^c, \quad t \geq 2. \quad (4)$$

The last term in equation (4) reflects the fact that the central bank's holdings of government bonds effectively earn a zero nominal rate of interest and thus a gross real return of p_{t-1}/p_t between t – 1 and t, whereas debt held by the public (the second term on the right) earns a positive nominal rate of interest.

2.3 Bank Behavior

As indicated above, at date t each young agent deposits the entire value of his after-tax endowment, $\omega - \tau_t$, in a bank. Banks use these deposits to acquire currency (cash reserves), bonds, and storage. They behave competitively in asset markets, taking as given the gross real rate of return on reserves (p_t/p_{t+1}), on bonds (R_t), and on storage (x). In issuing liabilities, banks can be thought of as coalitions of ex ante identical young agents that choose to pay a gross real rate of return per unit deposited of d_t^m to agents who relocate between t and t + 1 and a return of d_t^n to agents who remain settled.

A representative bank faces three constraints. It faces the balance-sheet constraints

$$m_t + b_t^p + s_t \leq \omega - \tau_t, \quad t \geq 1, \quad (5)$$

$$\left[\pi d_t^m + (1-\pi)d_t^n\right](\omega - \tau_t) \leq m_t\frac{p_t}{p_{t+1}} + xs_t + R_t b_t^p, \quad t \geq 1, \quad (6)$$

where m_t denotes the bank's holdings of real cash reserves per depositor at t, b_t^p denotes its bond holdings in real terms per depositor, and s_t denotes its storage investments in real terms per depositor. Equation (5) constrains bank assets to not exceed bank liabilities, while equation (6) asserts that total payments to depositors cannot exceed a bank's earnings from its asset holdings. In addition, given current and future price levels, the bank's cash reserves must be sufficient to allow the bank to pay the return promised to young agents who relocate. This requires that

$$\pi d_t^m(\omega-\tau_t) \le m_t(p_t/p_{t+1}). \tag{7}$$

As a coalition of ex ante identical agents at *t*, a bank's problem is to maximize

$$\pi u\left[d_t^m(\omega-\tau_t)\right]+(1-\pi)u\left[d_t^n(\omega-\tau_t)\right],$$

subject to (5) through (7). If private agents can sell government bonds short, then an absence of arbitrage opportunities requires

$$R_t \ge x, \quad t \ge 1. \tag{8}$$

Throughout this paper, the focus is on situations where some goods storage occurs, so that (8) holds with equality at each date. In addition, if the gross nominal interest rate is defined by $I_t \equiv R_t(p_{t+1}/p_t) = x(p_{t+1}/p_t)$, then it is easy to verify that (7) holds with equality when $I_t > 1$ since banks will not want to carry cash reserves between periods. In what follows, the focus is on equilibria with $I_t > 1$ for all *t*.[2] When this condition is satisfied, the solution to the bank's maximization problem can be described as follows.

A representative bank's reserve-deposit ratio is denoted γ_t, where $\gamma_t \equiv m_t/(\omega-\tau_t)$. It is easy to verify that the bank's optimal reserve-deposit ratio is given by the following expression:

$$\gamma_t = \frac{1}{1+\left(\frac{1-\pi}{\pi}\right)I_t^{\frac{1-\rho}{\rho}}} \equiv \gamma(I_t). \tag{9}$$

The following lemma summarizes some important properties of the function $\gamma(I)$.

Lemma 1. The function $\gamma(I)$ has the following properties:

(a) $\gamma(1)=\pi$.

(b) $I\gamma'(I)/\gamma(I) = -((1-\rho)/\rho)\left[1-\gamma(I)\right]$.

In particular, $\gamma'(I)<0$ since $\rho \in (0,1)$. That is, higher nominal interest rates imply a higher opportunity cost of holding reserves and thus a lower reserve-deposit ratio for the bank.

Given the bank's optimal reserve-deposit ratio, equations (6) and (7) imply that

[2] See Paal and Smith (2000) and Smith (2001) for a discussion of closely related environments where positive nominal rates of interest are optimal.

$$d_t^m = \frac{\gamma(I_t)\left(\frac{p_t}{p_{t+1}}\right)}{\pi} = \frac{x\gamma(I_t)}{\pi I_t}$$

and

$$d_t^n = \frac{x\left[1-\gamma(I_t)\right]}{1-\pi}.$$

It follows that $d_t^n / d_t^m = I_t^{1/\rho}$. Thus, the higher the nominal rate of interest, the less insurance banks provide against the event of relocation. Intuitively, this is because relocated agents require cash, and the higher the nominal interest rate, the higher is the opportunity cost of holding cash reserves to finance the consumption of those who are relocated.

Finally, the maximized expected utility of a representative depositor at *t*, $V(I_t)$, is

$$V(I_t) \equiv \frac{(\omega-\tau_t)^{1-\rho}}{1-\rho}\left\{\pi^\rho\left[\frac{x\gamma(I_t)}{I_t}\right]^{1-\rho} + (1-\pi)^\rho\left[x\left(1-\gamma(I_t)\right)\right]^{1-\rho}\right\},$$

and $V'(I_t) < 0$. Welfare is decreasing in the nominal interest rate because of the less complete provision of insurance at higher interest rates.

The remainder of this paper characterizes the equilibria of this economy. To accomplish this, it is necessary to take a stand on how monetary and fiscal policy are conducted. In essence, the issue is whether the treasury or the central bank acts first.[3] Following Leeper 1991, the government entity taken to act first is said to behave actively, while the one that reacts is said to behave passively. The next two sections consider alternative scenarios that differ in terms of which entity acts first in setting policy.

3. Equilibria with an Active Treasury and Passive Central Bank

This section considers a scenario in which the treasury exogenously sets the time path for the real value of government debt outstanding. This scenario seems most consistent with recent policy discussions about the constraints imposed on a central bank by a shrinking stock of government debt. In particular, these discussions presume that the government debt will shrink

[3] This same issue arises in the "unpleasant monetarist arithmetic" literature. See Sargent and Wallace (1981).

in a way that is outside the control of the central bank, and that the central bank must adjust its behavior accordingly.

Clearly the bond market clears if $b_t^p + b_t^c \equiv b_t = B_t / p_t$. Thus, to capture the desired scenario, it is assumed that $\tau_t = \tau$ and $g_t = g$ for all *t*, with τ given exogenously, and that the treasury sets a target time path for the stock of government debt of

$$b_t = \mu b_{t-1} + (1-\mu)\bar{b}, \quad t \geq 1, \tag{10}$$

$b_0 \geq 0$ given. In (10), $\bar{b} > 0$ is the real value of outstanding government debt in a steady state and $\mu \in (0,1)$. Given these assumptions about the time path of the stock of government debt, a competitive equilibrium can be characterized.

In equilibrium, government bonds held by the public must compete with storage in private portfolios. Hence the gross real rate of return on government debt, R_t, must equal *x*. As a result, the treasury's budget constraint (4) can be written as

$$g - \tau = b_t^p + b_t^c - x b_{t-1}^p - \frac{p_{t-1}}{p_t} b_{t-1}^c, \quad t \geq 2. \tag{11}$$

Equation (11), combined with the central bank's balance-sheet constraint, (1), yields the standard consolidated balance sheet for the treasury and the central bank:

$$g_t - \tau_t = b_t^p + \frac{M_t}{p_t} - \frac{M_{t-1}}{p_{t-1}} \frac{p_{t-1}}{p_t} - x b_{t-1}^p, \quad t \geq 2.$$

This can be rewritten using the definitions $p_{t-1}/p_t \equiv x/I_{t-1}$ and $b_t \equiv b_t^p + b_t^c$ as

$$g - \tau = b_t - x b_{t-1} + x b_{t-1}^c \left(\frac{I_{t-1} - 1}{I_{t-1}} \right), \quad t \geq 2. \tag{12}$$

Similarly, the money market clears if

$$b_t^c = \frac{M_t}{p_t} = m_t \equiv \gamma(I_t)(\omega - \tau), \quad t \geq 1, \tag{13}$$

since all beginning-of-period demand for real balances derives from the reserve demand of banks. Together, equations (12) and (13) imply that *any* equilibrium must satisfy

$$g - \tau = b_t - x b_{t-1} + x(\omega - \tau)\gamma(I_{t-1}) \left(\frac{I_{t-1} - 1}{I_{t-1}} \right), \quad t \geq 2. \tag{14}$$

3.1 *Steady States*

3.1.1 Candidate Steady States

In a steady state, $b_t = b_{t-1} = \bar{b}$ and $I_t = I_{t-1} = I$. Hence, (14) reduces to

$$\bar{b} = \frac{1}{x-1}\left(\tau - g + x(\omega-\tau)\gamma(I)\frac{I-1}{I}\right). \tag{15}$$

Equation (15) states that interest payments on the stock of debt outstanding must be financed by a combination of primary budget surpluses ($\tau - g$) and seigniorage revenue $\left(x(\omega-\tau)\gamma(I)\left[(I-1)/I\right]\right)$. Defining the function $H(I) \equiv \gamma(I)(I-1)/I$, which is approximately seigniorage per unit of deposits, allows seigniorage revenue to be written as $\left(x(\omega-\tau)H(I)\right)$ and (15) to be written as

$$\bar{b} = \frac{1}{x-1}\left[\tau - g + x(\omega-\tau)H(I)\right] \equiv \psi(I). \tag{16}$$

Clearly, $\psi(I)$ is proportional to the sum of seigniorage revenue and the primary budget surplus.

The properties of $H(I)$ are stated in the following lemma, which is proven in the appendix. The lemma implies that the relationship between the nominal interest rate and seigniorage revenue follows a standard inflation-tax Laffer curve.

Lemma 2. The function $H(I)$ has the following properties:

(a) $H(1) = 0$.

(b) $\lim_{I\to\infty} H(I) = 0$.

(c) $IH'(I)/H(I) = \left[1/(I-1)\right] - \left[(1-\rho)/\rho\right]\left[1-\gamma(I)\right]$.

(d) $H'(I) \geq 0$ holds iff $I \leq \hat{I}$, where $\hat{I} > 1$ is defined by

$$1 \equiv \left(\frac{1-\rho}{\rho}\right)\left(\hat{I}-1\right)\left[1-\gamma\left(\hat{I}\right)\right]. \tag{17}$$

It follows that equilibrium condition (16) can be depicted as in Figure 2 below for the case $\tau - g < 0$. From (16) and Figure 2, the next result is apparent.

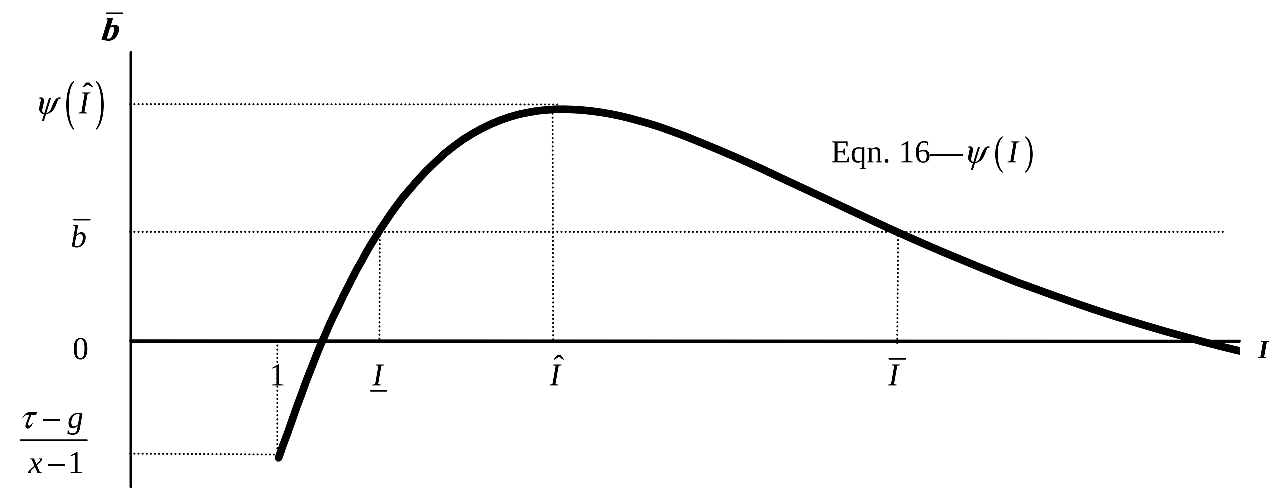

Figure 2 — The Government Budget Constraint in Equilibrium

Proposition 1. The potential for steady-state equilibria to exist is as follows:

(a) Suppose that

$$\bar{b} \in \left(\frac{\tau - g}{x-1}, \left[\frac{1}{x-1}\right]\left[\tau - g + x(\omega - \tau) H(\hat{I})\right]\right). \tag{18}$$

Then there are exactly two candidate steady-state values of the gross nominal interest rate. These are $\underline{I}$ and $\bar{I}$ in Figure 2.

(b) If $\bar{b} < (\tau - g)/(x-1)$ or $\bar{b} > \frac{1}{x-1}\left[\tau - g + x(\omega-\tau) H(\hat{I})\right]$, then no steady-state equilibrium exists.[4]

It follows that a steady state with a positive nominal interest rate exists only if

$$(x-1)\bar{b} > \tau - g. \tag{19}$$

Condition (19) implies $\tau - \left[g + (x-1)\bar{b}\right] < 0$, so if the treasury had to pay the market interest rate on all outstanding debt—including that held by the central bank—then it would run a deficit. Equation (19) is consistent with the existence of a primary government budget surplus (g < τ), or

[4] This article considers only equilibria with $I > 1$. However, it is also possible to show that generically there are no equilibria with $I = 1$ under the conditions stated.

with a surplus when debt held by the central bank effectively does not earn interest (i.e., when the central bank rebates interest earnings on its bond holdings to the treasury). In what follows, (18) is assumed to hold so that it is potentially feasible for the treasury to finance its primary budget deficit and the interest payments on its outstanding government debt of $\bar{b}$. However, as will be shown below, some projections for government budget surpluses and debt levels violate (19), making them inconsistent with the existence of a steady-state equilibrium.

Intuitively, the factors that determine candidate steady-state nominal rates of interest are the same factors at play in conventional analyses of seigniorage Laffer curves (Sargent, 1987, chapter 7; Azariadis, 1993, chapter 19). The value x/I is the gross real rate of return on real balances in a steady state. If the government budget deficit (inclusive of interest payments on the government debt if all debt earns the market real return) is not too large, there is more than one value of the rate of return on real balances potentially consistent with a steady-state equilibrium. The assumption that (18) holds implies the existence of exactly two such values, $x/\underline{I}$ and $x/\bar{I}$. The steady state with $I_t = \underline{I}$ $(\bar{I})$ thus has a relatively low (high) associated rate of inflation. And $V(\underline{I}) > V(\bar{I})$, so values of $I_t < (>) \hat{I}$ are on the good (bad) side of the Laffer curve. In other words, the candidate equilibrium with $I_t = \underline{I}$ is Pareto superior to the equilibrium with $I_t = \bar{I}$.

3.1.2 Steady States Permitted by Limits on the Stock of Debt Outstanding

Some, or all, of the candidate steady-state equilibria might not constitute legitimate equilibria because the central bank can at most hold all of the government debt outstanding. As a result, in addition to (16), a steady state must have $b^c \le \bar{b}$. In conjunction with the central bank's balance sheet, this constraint requires that $M_t/p_t = m \le b^c \le \bar{b}$, which implies

$$(\omega - \tau)\gamma(I) \le \bar{b}. \qquad (20)$$

Condition (20), together with the fact that $\gamma(I)$ is decreasing in the nominal interest rate, has two implications. First, for a given target level of government debt ($\bar{b}$), it implies a lower bound on the steady-state equilibrium interest rate because $\bar{b}$ strictly limits the amount of liquidity the central bank can inject into the economy. At a nominal interest rate that is too low to satisfy (20), private demand for liquidity exceeds the amount of liquidity the central bank can supply. This implies that, in addition to satisfying (16), a steady-state equilibrium must not lie below the bound implied by (20) in Figure 3 below (i.e., it must be in the shaded area in the figure).

Second, there are three possible outcomes regarding the existence of steady states.

Case 1: $(\omega-\tau)\gamma(\underline{I}) \leq \bar{b}$. In this case, both candidate steady states satisfy (20). Essentially, there is enough government debt outstanding for the central bank to supply the liquidity required even at the steady state with the low nominal interest rate (low inflation rate). This is the case illustrated in Figure 3.

Case 2: $(\omega-\tau)\gamma(\underline{I}) > \bar{b} \geq (\omega-\tau)\gamma(\bar{I})$. Here the quantity of government debt outstanding is too low to allow the central bank to meet private demands for liquidity at the steady state with the low nominal interest rate (low inflation rate). As a result, there is a unique steady-state equilibrium at the high nominal interest rate, $\bar{I}$.

Case 3: $(\omega-\tau)\gamma(\bar{I}) > \bar{b}$. When this case arises, the stock of government debt outstanding is so low that the central bank cannot supply the liquidity demanded even at

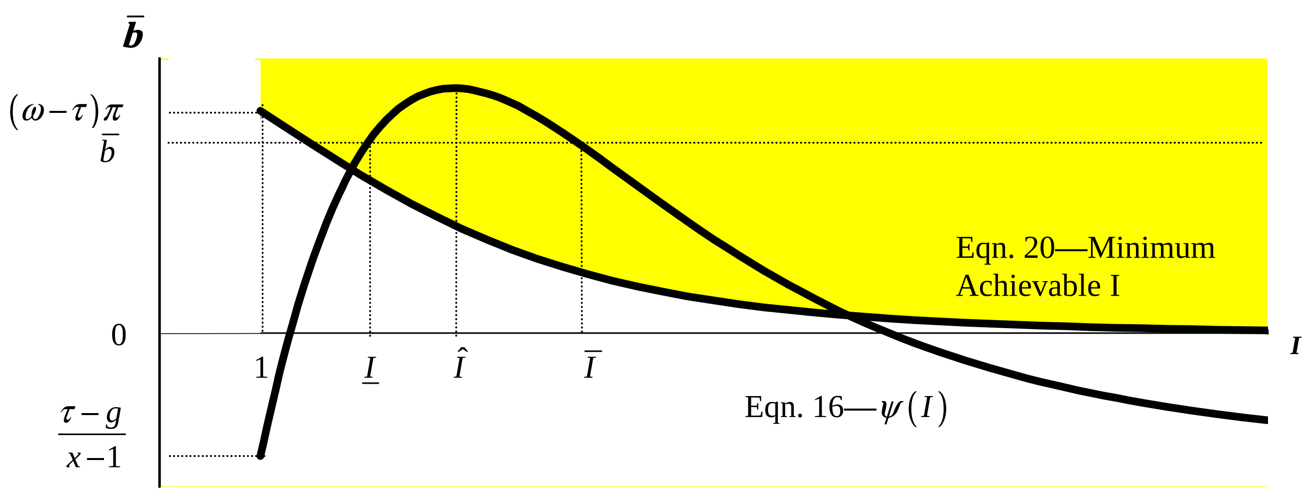

Figure 3 — Determination of a Steady-State Equilibrium When the Level of Government Debt Outstanding is Exogenous

the candidate steady state with the high nominal interest rate (high inflation rate). Consequently, there is no steady-state equilibrium.

Clearly cases 2 and 3 are problematic in the following sense. In case 2 the nominal interest rate and inflation rate are higher than they would need to be if only the treasury issued more debt. Also, steady-state welfare is lower than it needs to be. Thus there is a clear sense in which the outstanding stock of debt is too low. Case 3 is even more problematic, and there is an even stronger sense in which the stock of debt outstanding is too small.

It remains to describe when cases 1, 2, and 3 obtain. With $\bar{b}_i$ denoting the smallest value of $\bar{b}$ for which case *i* obtains, $i = 1, 2$, it is easy to see that case 1 results iff

$$\gamma^{-1}\left(\frac{\bar{b}}{\omega-\tau}\right) < \hat{I}, \tag{21}$$

$$\bar{b} \geq \frac{1}{x-1}\left[\tau - g + x(\omega-\tau)H\left(\gamma^{-1}\left(\frac{\bar{b}}{\omega-\tau}\right)\right)\right]. \tag{22}$$

These conditions are satisfied for $\bar{b} \in \left[\bar{b}_1, \hat{\bar{b}}\right)$, as shown in Figure 4 below. Similarly, case 2 obtains if

$$\bar{b} < \frac{1}{x-1}\left[\tau - g + x(\omega-\tau)H\left(\gamma^{-1}\left(\frac{\bar{b}}{\omega-\tau}\right)\right)\right] \tag{23}$$

or if $\gamma^{-1}\left(\bar{b}/(\omega-\tau)\right) \geq \hat{I}$ and $\bar{b} = \frac{1}{x-1}\left[\tau - g + x(\omega-\tau)H\left(\gamma^{-1}\left(\frac{\bar{b}}{\omega-\tau}\right)\right)\right]$. Figure 4 illustrates that values of $\bar{b} \in \left[\bar{b}_2, \bar{b}_1\right)$ satisfy these conditions. Finally, the economy is in case 3 if $\gamma^{-1}\left[\bar{b}/(\omega-\tau)\right] \geq \hat{I}$ and $\bar{b} > \frac{1}{x-1}\left[\tau - g + x(\omega-\tau)H\left(\gamma^{-1}\left(\frac{\bar{b}}{\omega-\tau}\right)\right)\right]$. Values of $\bar{b} \in \left[(\tau-g)/(x-1), \bar{b}_2\right)$ satisfy this case.

3.1.3 Budget Surpluses and the Existence of Steady States

As is apparent from Figure 4, $\bar{b}_1$ and $\bar{b}_2$ are the stocks of outstanding debt that satisfy conditions (16) and (20) simultaneously and with equality. Clearly, they depend on the size of the government's budget surplus. Specifically, as the government surplus becomes larger, the

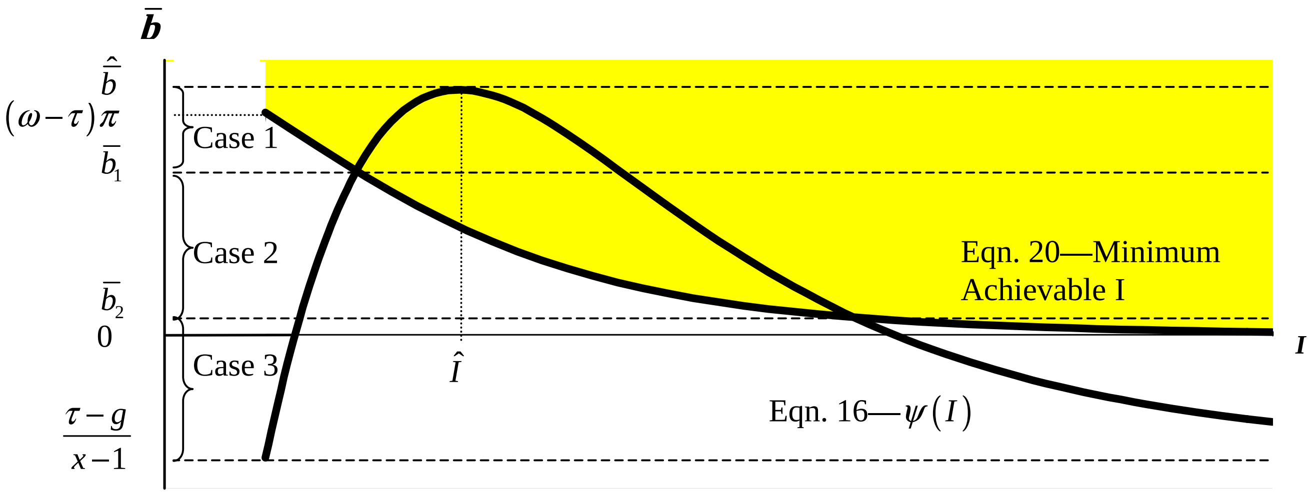

Figure 4—The Relationship Between the Stock of Outstanding Debt and Existence of Steady-State Equilibria

Laffer Curve shifts up, altering the intersection of conditions (16) and (20) and thus $\bar{b}_1$ and $\bar{b}_2$. And when $\bar{b}_1$ and $\bar{b}_2$ change, the scope for cases 1 through 3 to arise also changes.

It remains to determine how $\bar{b}_1$ and $\bar{b}_2$, and thus the regions in which cases 1 through 3 apply, depend on $\tau - g$. From Lemmas 1 and 2 it follows that there are at most two intersections of (16) and (20). The value $\bar{b}_1$ $\left(\bar{b}_2\right)$ is the largest (smallest) solution to

$$\bar{b} = \frac{1}{x-1}\left[\tau - g + x(\omega-\tau)H\left(\gamma^{-1}\left(\frac{\bar{b}}{\omega-\tau}\right)\right)\right]. \tag{24}$$

Since any solution to (24) must be a candidate steady state, $\bar{b}_1$ and $\bar{b}_2$ must lie in the interval $\left[(\tau-g)/(x-1),(1/(x-1))\left(\tau-g+x(\omega-\tau)H\left(\hat{I}\right)\right)\right]$, according to Proposition 1. Debt level $\bar{b}_1$ must also satisfy $\gamma^{-1}\left(\frac{\bar{b}_1}{\omega-\tau}\right) \leq \hat{I}$, while $\bar{b}_2$ must satisfy $\gamma^{-1}\left(\frac{\bar{b}_2}{\omega-\tau}\right) > \hat{I}$. It follows that there are four possible outcomes, as illustrated in Figure 5 below, regarding the existence of values of $\bar{b}_1$ and $\bar{b}_2$ satisfying these conditions and thus the existence of cases 1 through 3. Proposition 2 states this result.

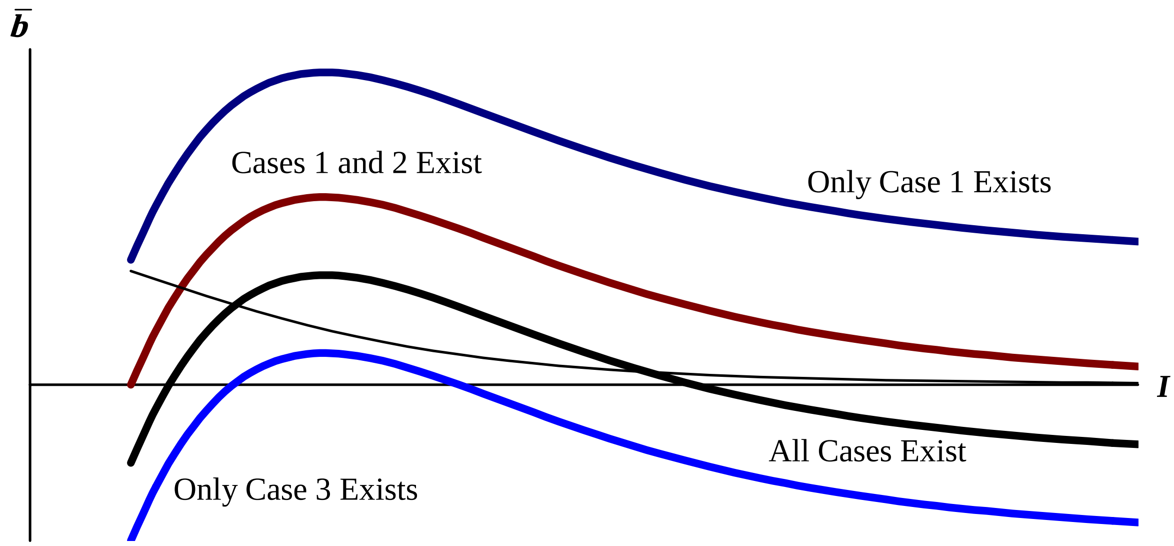

Figure 5—The Effects of Government Surpluses on the Existence of Steady-State Equilibria

Proposition 2. Given $\bar{b} \in \left[(\tau - g)/(x-1), (1/(x-1))\left(\tau - g + x(\omega - \tau) H(\hat{I})\right) \right]$, the state of the government's budget has the following implications for the existence of steady-state equilibria:

(a) Suppose that $\tau \geq g$.

(i) If $(\omega - \tau)\gamma(1) \leq \frac{1}{x-1}\left[\tau - g + x(\omega - \tau) H(1)\right]$, then (24) has no solution and case 1 obtains for any value of $\bar{b}$.

(ii) If $(\omega - \tau)\gamma(1) > \frac{1}{x-1}\left[\tau - g + x(\omega - \tau) H(1)\right]$ but $(\omega - \tau)\gamma(\hat{I}) < \frac{1}{x-1}\left[\tau - g + x(\omega - \tau) H(\hat{I})\right]$, then (24) has only one solution: $\bar{b}_1$. It follows that case 1 obtains if $\bar{b} \geq \bar{b}_1$ and case 2 obtains otherwise.

(b) Suppose that $\tau < g$, and let $\tilde{I}$ be the value of the nominal interest rate that satisfies $x = ((1-\rho)/\rho)(\tilde{I} - x)\left[1 - \gamma(\tilde{I})\right]$. If $(\omega - \tau)\gamma(\tilde{I}) < \frac{1}{x-1}\left[\tau - g + x(\omega - \tau) H(\tilde{I})\right]$, then (24) has two solutions, $\bar{b}_1$ and $\bar{b}_2$. The economy is in case 1 if $\bar{b} \geq \bar{b}_1$, case 2 if $\bar{b}_1 < \bar{b} \leq \bar{b}_2$, and case 3 otherwise. If instead $(\omega - \tau)\gamma(\tilde{I}) >$

$\frac{1}{x-1}\left[\tau - g + x(\omega-\tau)H\left(\tilde{I}\right)\right]$, then (24) has no solution. Equation (20) is violated and the economy is in case 3 for all $\bar{b}$.

The proof of Proposition 2 appears in the appendix. Part (a) of the proposition has an interesting implication: if there is a primary government budget surplus, then existence of a steady-state equilibrium is guaranteed because case 3 never obtains. With a primary government budget surplus, the central bank can always provide enough liquidity to support at least the equilibrium with a high interest rate (high inflation rate). Part (b) of the proposition states the obverse: if the primary government budget deficit is too large, a central bank might be unable to provide enough liquidity to support any equilibrium. This can occur even if $\bar{b} \in \left[(\tau-g)/(x-1), [1/(x-1)]\left[\tau - g + x(\omega-\tau)H\left(\hat{I}\right)\right]\right]$.

3.1.4 The Effects of a Reduction in the Steady-State Stock of Treasury Debt

The consequences of a decision by the treasury to reduce the real value of its outstanding debt depend on which of the cases discussed above obtains. Assuming the stock of debt outstanding is initially $\bar{b}'$, putting the economy in case 1, then one possibility is that the economy remains in case 1 after the reduction in the debt. As depicted in Figure 6 below, this could come about if the debt fell to $\bar{b}''$. Two steady states exist both before and after the reduction in the stock of debt. The nominal rate of interest and the inflation rate fall (rise) on the good (bad) side of the Laffer curve as debt is reduced from $\bar{b}'$ to $\bar{b}''$. Since the nominal interest rate falls to $\underline{I}''$ on the good side of the Laffer curve, real balances rise, as does the money-bond ratio $m/\bar{b} = \gamma(I)(\omega-\tau)/\bar{b}$. The corresponding reduction in inflation on the good side of the Laffer curve thus reflects conventional unpleasant-monetarist-arithmetic arguments (Sargent and Wallace, 1981; Bhattacharya, Guzman, and Smith, 1998).

A second possibility is that the stock of debt is reduced to $\bar{b}'''$, pushing the economy from case 1 to case 2, as illustrated in Figure 6. Whether the initial equilibrium was at $\underline{I}'$ or $\bar{I}'$, the

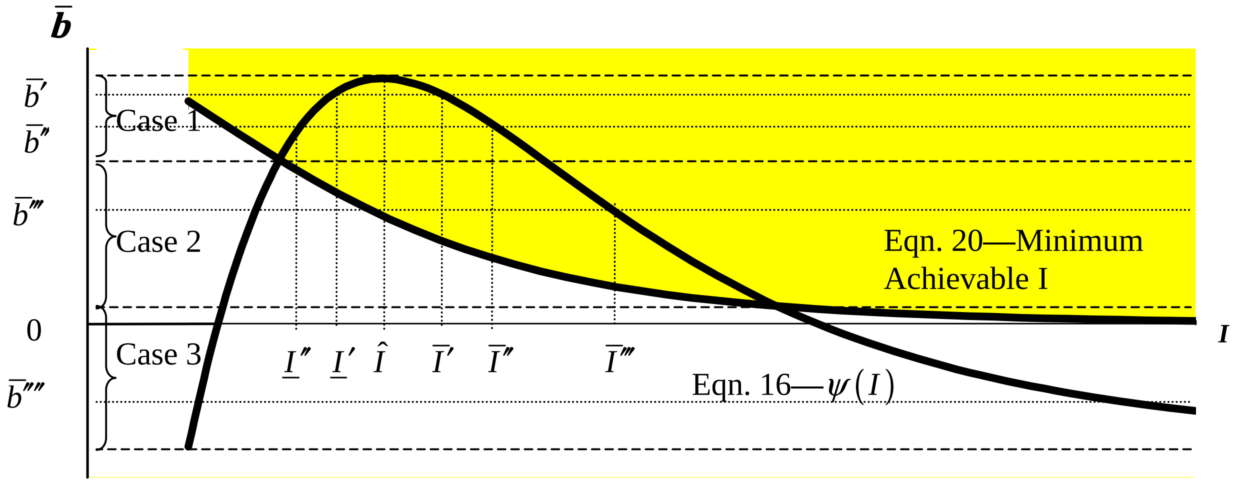

Figure 6—The Effects of a Reduction in the Steady-State Stock of Treasury Debt

steady-state equilibrium shifts to $\bar{I}'''$. That is, a reduction in the stock of debt outstanding that drives the economy into case 2 is necessarily associated with an increase in the nominal interest rate and the inflation rate. And if the economy was initially on the Pareto superior side of the Laffer curve, say at $\underline{I}'$, the resulting increase in inflation and the nominal interest rate could be quite large. Likewise, if the economy is in case 2 before and after the reduction in the debt, then the decline in $\bar{b}$ must result in a higher steady state nominal interest rate and inflation rate. It is straightforward to show that the reduction in debt is associated with a reduction in the ratio of money to bonds. Since the rate of inflation rises, a result analogous to the unpleasant-monetarist-arithmetic result obtains. This is the case despite the economy's being on the wrong side of the Laffer curve.

Finally, if the stock of debt falls sufficiently far, say to $\bar{b}''''$ in Figure 6, so that the economy is pushed into case 3, then an equilibrium ceases to exist. Of course this can only occur if case 3 can exist, which requires that there be a primary government budget deficit.

3.2 Dynamics

This section considers the kinds of equilibrium paths that can be observed outside of steady states. Combining equations (14) and (10) yields the condition that determines the nominal interest rate at any date:

$$\frac{g-\tau-(1-\mu)\bar{b}+(x-\mu)b_{t-1}}{x(\omega-\tau)}=\gamma(I_{t-1})\left(\frac{I_{t-1}-1}{I_{t-1}}\right)\equiv H(I_{t-1}). \tag{25}$$

And the constraint that $b_{t-1}^{c}=\gamma(I_{t-1})(\omega-\tau)\leq b_{t-1}$ requires that

$$I_{t-1}\geq\gamma^{-1}\left(\frac{b_{t-1}}{\omega-\tau}\right) \tag{26}$$

In what follows, attention is restricted to situations in which $b_0\geq\bar{b}$ and thus $b_{t-1}\geq\bar{b}$ holds for all $t \geq 1$.

Since (25) involves only terms dated $t - 1$, the only intrinsic dynamics governing the evolution of the nominal interest rate are embodied in the dynamics governing the evolution of the debt (equation (10)). Alternatively, the determination of the nominal interest rate at $t - 1$ is independent of the determination of the nominal rate at any other date.[5] This fact gives rise to several possibilities regarding the types of dynamical equilibria that can be observed. Some examples follow.

Example 1: Suppose that $\bar{b}\geq\bar{b}_1$. Then if b_0 is not too large, (25) will have two solutions, both satisfying (26). In other words, case 1 will obtain at all dates. Letting $\underline{I}(b_{t-1})$ $\left(\bar{I}(b_{t-1})\right)$ denote the smallest (largest) solution to (25), it follows that the only restriction on equilibrium sequences $\{I_t\}$ is that $I_t\in\{\underline{I}(b_t),\bar{I}(b_t)\}$ for all t. That is, there can be equilibrium sequences like $\{\underline{I}(b_1),\bar{I}(b_2),\underline{I}(b_3),\bar{I}(b_4),...\}$. Of course, many such sequences with different patterns of fluctuations in nominal rates of interest and the inflation rate can be observed. In addition, equilibrium sequences need not converge; fluctuations need not dampen asymptotically. Thus, in a case 1 economy, severe indeterminacies arise.

[5] This is quite different from the standard situation that arises in conventional pure-exchange, overlapping-generations models with an exogenously given government budget deficit (e.g., Sargent, 1987, chapter 7, or Azariadis 1993, chapter 19).

This observation illustrates a potential tension between the efficiency and the determinacy of equilibrium with respect to the choice of $\bar{b}$.[6] In particular, the attainment of a steady state with a low nominal interest rate (low inflation rate) requires that $\bar{b}$ be large enough so that the economy is in case 1. But, if it is, there will be a large set of equilibrium paths, as just shown.

Example 2: Suppose that $g > \tau$ and that $\bar{b}_1 > \bar{b} \geq \bar{b}_2$. Then with respect to steady states, the economy is in case 2. For *t* sufficiently large, (25) has only one solution that satisfies (26). Or, in other words, for *t* sufficiently large, there is a unique equilibrium nominal rate of interest. Of course, if b_0 is large enough, there may be a finite number of periods where the nominal interest rate can take on either of two values.

3.3 Stronger Restrictions on Central Bank Holdings of Government Debt

In practice, the central bank may face stronger restrictions than just $b_t^c \leq b_t$. For example, current rules governing the Federal Reserve's System Open Market Account put limits on the fraction of outstanding debt that can be held of various maturities. These limits range from 35 percent of the outstanding bills and coupons with a maturity of less than one year to 15 percent for issues with maturities of 10 years or more (Dudley and Youngdahl, 2000, p. 4).

Such restrictions imply that the central bank's debt holdings must satisfy the stronger constraint $b_t^c \leq \theta b_t$, given some exogenously set value $\theta \in (0,1)$. With this restriction, the analysis of steady states requires that (20) be replaced with

$$(\omega - \tau)\gamma(I) \leq \theta\bar{b}\ . \tag{27}$$

The findings above still apply, except that case 1 obtains iff $\gamma^{-1}\left(\theta\bar{b}/(\omega-\tau)\right) < \hat{I}$ and

[6] Smith (1991, 1994) and Woodford (1994) also explore the tensions between efficiency and determinacy of equilibrium.

$$\bar{b} \geq \left(\frac{1}{x-1}\right)\left(\tau - g + x(\omega-\tau)H\left[\gamma^{-1}\left(\frac{\theta\bar{b}}{\omega-\tau}\right)\right]\right). \tag{28}$$

Similarly, case 2 obtains if

$$\bar{b} < \left(\frac{1}{x-1}\right)\left(\tau - g + x(\omega-\tau)H\left[\gamma^{-1}\left(\frac{\theta\bar{b}}{\omega-\tau}\right)\right]\right). \tag{29}$$

and so on.

Figure 7 shows how a reduction in θ shifts (27) up, raising $\bar{b}_1$ and reducing the range of debt levels consistent with the existence of case 1. It considers high and low values of θ (the low one corresponding to a more binding limitation), and assumes that the stock of debt outstanding is $\bar{b}(\theta_H)$, so that the economy is in case 1 with the less binding restriction, θ_H. When θ is reduced to θ_L, the smallest stock of debt consistent with the existence of case 1 rises to $\bar{b}_1(\theta_L)$. The economy is driven to case 2 and thus a Pareto inferior equilibrium. Moreover, if $\tau < g$, even tighter restrictions than those shown could push the economy into case 3. Thus, restrictions on the amount of treasury debt that can be held by the central bank can reduce

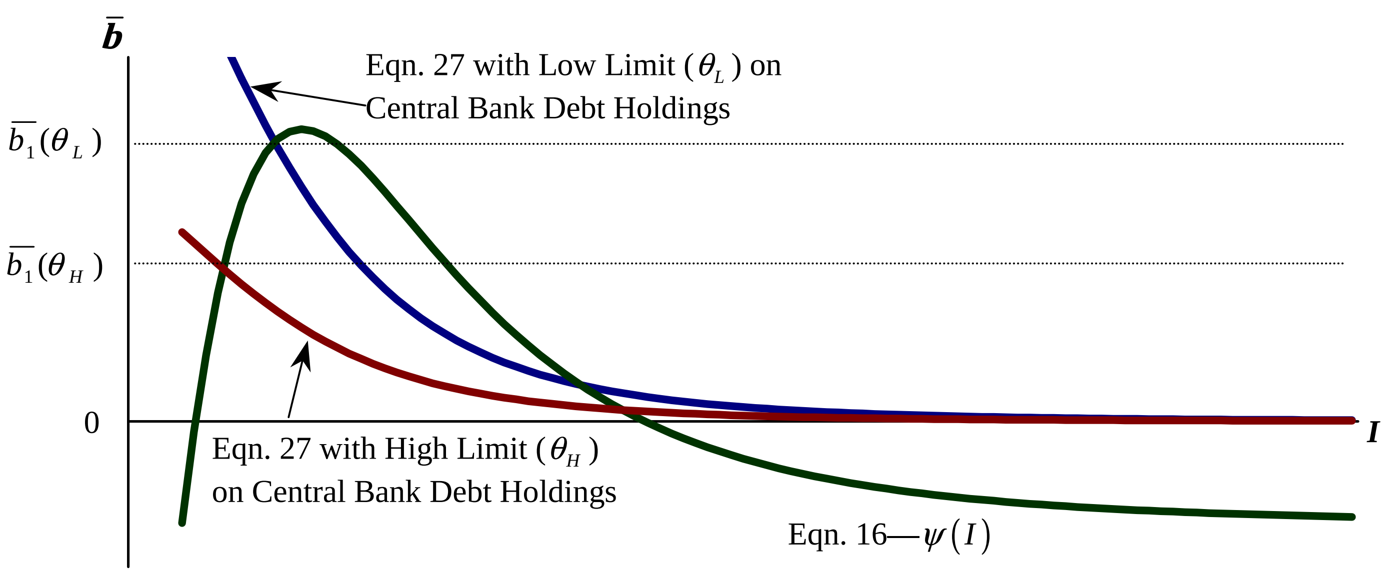

Figure 7—The Effect of Restrictions on the Share of Debt Held by the Central Bank

welfare by moving the economy from case 1 to case 2, or can pose problems for the existence of equilibrium by moving the economy to case 3. The model suggests, therefore, that there is no economic justification for such restrictions, and there is substantial justification for removing them if they are in place.

3.4 How Much Debt Does the Central Bank Need?

It is possible to attach empirically plausible parameter values to the model. The term $\bar{b}_1$ can be calculated to obtain an estimate of how much debt must be issued, given the primary budget surplus, to ensure that case 1 obtains.

According to the 2001 *Economic Report of the President* (Table B-78), the government surplus in 2000 was \$236.2 billion. From Table B-69, total debt (L) outstanding averaged \$17,810 billion during the first 10 months of 2000.[7] This broad monetary aggregate corresponds well to deposits in the model since the model assumes that all holding of government liabilities (except debt held by the central bank) and all investment is done by intermediaries. Total federal debt held by the public (Table B-78) was \$3410.1 billion in 2000. The average net annual nominal rate of interest on treasury securities with maturities of three years or longer (Table B-73) was about 6 percent in 2000. In addition, conventional estimates give an annual real rate of return of 1.04 (Prescott 1986).

In calibrating the model, the question arises of how to interpret the length of a period. For simplicity, the analysis assumes that all government debt is repaid one period after issue. Therefore, the length of a period is taken to be the average maturity of the federal debt, which is about six years (*Economic Report of the President*, Table B-88). Compounded over the six years until the average treasury security matures, the rates of return reported above imply that reasonable values for I and x are 1.42 and 1.27, respectively.

Ideally, parameter choices would approximately match two additional observations. First, $I\gamma'(I)/\gamma(I) = -\left((1-\rho)/\rho\right)\left(1-\gamma(I)\right)$, the long-run interest elasticity of excess reserves,

[7] L consists of debt outstanding of U.S. federal, state, and local governments and the private nonfinancial sector.

should be consistent with empirical estimates of this elasticity.[8] According to Goldfeld (1966, p. 149), this elasticity is about –0.3. Staff of the Board of Governors of the Federal Reserve System report that this estimate of the long-run elasticity is still reasonable today. In the model, an elasticity of –0.3 is obtained by setting $\rho = 0.75$. This elasticity exhibits little sensitivity to any reasonable choice of π.

Second, the observed debt level, budget surplus, and nominal interest rate should be consistent with the existence of an equilibrium when sustained as steady-state values (i.e., should satisfy (16)). Whether the model matches this observation depends primarily on the value used for π. For a debt level of \$3410 billion, a surplus of \$236.2 billion, deposits of \$17810 billion, $I = 1.42$, and $x = 1.27$, equation (16) requires $\gamma(1.42) = 0.10$. With $\gamma(I) = \left[1+\left((1-\pi)/\pi\right) I^{(1-\rho)/\rho}\right]^{-1}$, $\pi = 0.111$. Alternatively, one might like $\gamma(1.42)$ to match the observed ratio of base money to deposits (with deposits taken to be L). That ratio is 0.03. Clearly, this value is small relative to 0.1. To have $\gamma(1.42) = 0.03$, it is necessary to set $\pi = 0.034$. Interestingly, when $\pi = 0.034$, equation (16) is everywhere above equation (20) given the government surplus from the data. However, it is also true, given current debt levels, surpluses, and rates of return, that equation (16) has no solution when $\pi = 0.034$. That is, if $\pi = 0.034$, then the model implies that current debt service is too large to be financed. Therefore, since intuition provides little guidance in picking π, a value of $\pi = 0.034$ ($\pi = 0.111$) is taken to be a lower (upper) bound on π, and the value of $\bar{b}_1$ corresponding to both values of π is calculated.

Table 1 below reports estimates of $\bar{b}_1$ for the parameter values just described, as well as for different scenarios regarding government budget surpluses. Varying the size of the surplus has little impact on the minimum amount of debt required for case 1 to obtain.

Of course, these calculations presume that the Federal Reserve System can hold the entire government debt. As noted above, current operating procedures allow it to hold less than 35 percent of the debt (Dudley and Youngdahl, 2000, p. 4). To capture this, Table 1 also shows estimates of $\bar{b}_1$ assuming that the Federal Reserve can hold at most 30 percent ($\theta = 0.3$) of the

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[8] In the model, all reserves are excess reserves.

Government Surplus ($ billions)	Minimum Stock of Government Debt $\bar{b}_1$ ($ billions, $\rho = 0.75$)			
	$\pi = 0.034$		$\pi = 0.111$	
	$\theta = 1$	$\theta = 0.3$	$\theta = 1$	$\theta = 0.3$
500	1885*	1994*	1971	5483
236.2	890*	1760*	1909	5088
143.3**	602	1645	1884	4915
0	561	1368	1843	4584
-97.8**	525	Case 3[+]	1813	4279

* Equation (16) is always above equation (20) for $\pi = 0.034$ and surpluses of $500 billion and $236.2 billion. Thus, the debt levels given are the minimum level required for equation (16) to have a solution.
** A surplus of $143.3 billion is the average surplus from 1998 through 2000. A deficit of –$97.8 billion is the average deficit from 1995 through 1997.
[+] With a deficit of –$97.8 billion , equation (16) is always below equation (20), so no steady-state equilibrium exists.

Table 1—Minimum Debt Levels for Case 1 to Obtain

total outstanding debt. For $\pi = 0.111$ and a surplus of $236.2 billion, the minimum stock of debt needed for case 1 almost triples, rising to $5088 billion from $1909 billion. Clearly, restrictions on the share of debt holdings by the Federal Reserve can significantly impact how much debt must be outstanding for case 1 to obtain.

The Congressional Budget Office's January 2001 projections show government debt held by the public falling to $1714 billion by 2005 and to $900 billion by 2010. The estimates presented in Table 1 therefore suggest that if $\pi = 0.111$, then the economy will be in case 2 by 2005, even in the absence of restrictions on the amount of debt the Federal Reserve System holds. However, if $\pi = 0.034$, there is no immediate danger of leaving case 1. Since these values of π bracket the range of reasonable choices, the possibility of transiting into case 2 by 2005 should, at a minimum, be taken seriously. Moreover, given current self-imposed restrictions on the Fed's holdings of government bonds, even with $\pi = 0.034$, Table 1 indicates that the economy will be in case 2 within the next decade if the primary government budget remains in surplus. Setting $\pi = 0.111$ implies that the economy is already in case 2, a possibility that seems reasonable to discount. Reality, then, should lie somewhere between the estimates obtained with $\pi = 0.034$ and $\pi = 0.111$. Table 1 therefore strongly suggests that case 2 will soon obtain unless limitations on Federal Reserve bond holdings are relaxed. In addition, case 2 might soon obtain even if these limits are eliminated. For example, Goldman Sachs (Dudley and Youngdahl, 2000) estimates that case 2 will obtain by 2003 under current operating procedures.

4. Equilibria with a Passive Treasury and Active Central Bank

This section describes how the analysis of Section 3 would differ under the assumption that the central bank acts first. In what follows, it is assumed that the central bank sets a sequence of values for $\{I_t\}$ and that the level of outstanding treasury debt is endogenous. For simplicity, tax collections τ_t remain constant at τ and exogenously given.

If the central bank follows the simple policy of setting $I_t = I^*$ for all *t*, then equation (12) reduces to

$$b_t = xb_{t-1} + g - \tau - x(\omega - \tau) H\left(I^*\right). \tag{30}$$

In addition, $b_t^c \le b_t$ in equilibrium requires that

$$(\omega - \tau)\gamma\left(I^*\right) \le b_t\,. \tag{31}$$

Equation (30) generates a solution sequence $\{b_t^*\}$ satisfying (31) only if $g - \tau - x(\omega - \tau) H\left(I^*\right) < 0$. When this condition is satisfied, the difference equation (30) has the configuration depicted in Figure 8 below. The only candidate equilibrium has $b^* = b_t$ for all *t*, with

$$b^* = \frac{x(\omega - \tau) H\left(I^*\right) - (g - \tau)}{x - 1}.$$

Then $(\omega - \tau)\gamma\left(I^*\right) \le b_t^*$ is satisfied iff

$$\gamma\left(I^*\right)\left(\frac{I^*-x}{I^*}\right) \geq \frac{g-\tau}{\omega-\tau}. \tag{32}$$

Given the primary budget surplus, equation (32) represents a restriction on the nominal interest rate target that can be selected by the central bank.

Matters are only slightly different if the central bank follows some feedback rule for selecting the target value of the nominal interest rate at *t*. If, for example, the feedback rule takes the form $I_t = f\left(I_{t-1}\right)$, given $I_0 > 1$, if there is a unique value $I^* > 1$ satisfying $I^* = f\left(I^*\right)$, and if $\left|f'\left(I^*\right)\right| < 1$, then the steady state is exactly as described. Moreover, it is a saddle, so that there is no potential for indeterminacies. Of course, the exact nature of equilibrium dynamics depends on the properties of the feedback rule *f*. Matters could be more complicated if the central bank sets the nominal interest rate in a way that depends on the history of the outstanding stock of debt.

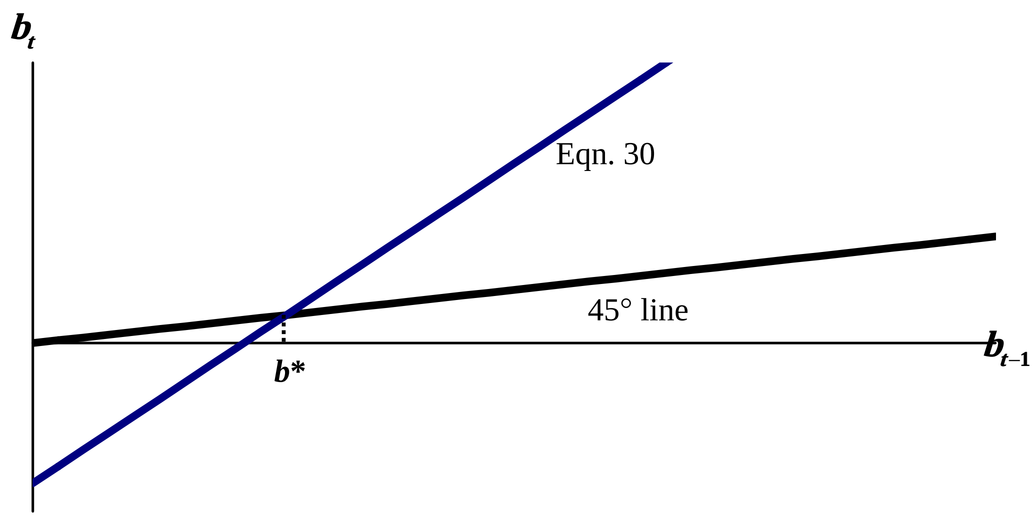

Figure 8—Law of Motion for the Stock of Debt, with *I* Exogenously Given

5. An Alternative: Conducting Monetary Policy Through the Discount Window

While monetary policy in the United States today is conducted almost entirely through open market operations, this is far from the only means by which it can be conducted. Early in its history, the Federal Reserve System added and withdrew money from the economy primarily by expanding and contracting discount-window lending. And in view of the fact that the Federal Reserve may soon hold as much government debt as the combination of its rules and the outstanding stock of debt allows, many are advocating that the Fed return to conducting monetary policy through the discount window (Minutes of the FOMC, 2001). This section discusses how this approach might work. It begins with a description of a central bank that conducts policy either via open market operations or through discount-window lending. It proceeds to consider how access to discount-window loans affects the behavior of banks. The section concludes with an analysis of equilibrium when a discount window operates.

5.1 Central Bank Operations with a Discount Window

As before, the central bank is assumed to hold treasury debt with a real per capita value of b_t^c at *t* and to rebate any interest income to the treasury. In addition, it is assumed that the central bank makes discount-window loans with a real value of ℓ_t^c per capita at *t* and charges a gross nominal interest rate (i.e., the discount rate) of I_t^c between *t* and *t* + 1 on those loans. Banks will only borrow at the discount window if $I_t^c \leq I_t$.[9] In what follows, the possibility that the discount rate is below the market rate is allowed. Thus, the discount rate serves as an additional instrument of monetary policy.[10]

In this environment, the central bank's balance sheet requires that

$$\frac{M_t}{p_t} \leq b_t^c + \ell_t^c. \tag{33}$$

In addition, since the central bank rebates all of its nominal interest earnings to the treasury, the real value of its rebates at *t* is given by

$$\left(x - \frac{p_{t-1}}{p_t}\right) b_{t-1}^c + \left(I_{t-1}^c - 1\right)\left(\frac{p_{t-1}}{p_t}\right) \ell_{t-1}^c.$$

[9] This aspect of the model derives from the fact that each bank's withdrawal demand is perfectly predictable. It would be interesting to consider how matters might change if banks confronted stochastic withdrawal demands.

[10] There has been some discussion in policy circles recently, given the relative inactivity of the discount window for the last 20 years, of the possible need for subsidies to induce banks to use the window more intensively if discount-window lending is to be a means of conducting monetary policy.

It follows that the treasury's budget constraint at date *t* is

$$g-\tau=\frac{B_t}{p_t}-R_{t-1}\frac{B_{t-1}}{p_{t-1}}+\left(x-\frac{p_{t-1}}{p_t}\right)b_{t-1}^c+\left(I_{t-1}^c-1\right)\frac{p_{t-1}}{p_t}\ell_{t-1}^c. \tag{34}$$

Using the central bank's balance sheet constraint in (34) yields the consolidated budget constraint of the treasury and the central bank:

$$g-\tau=\frac{B_t}{p_t}-R_{t-1}\left(\frac{B_{t-1}}{p_{t-1}}-b_{t-1}^c\right)-\frac{p_{t-1}}{p_t}b_{t-1}^c+I_{t-1}^c\frac{p_{t-1}}{p_t}\ell_{t-1}^c-\ell_t^c. \tag{35}$$

5.2 Bank Behavior in the Presence of a Discount Window

Bank behavior is exactly as described in Section 2, except that now banks have access to a discount window. It is assumed that the central bank imposes a limit on discount-window borrowing of $\ell_t \le \ell_t^c$, where ℓ_t denotes the amount borrowed from the discount window by a representative bank at *t*. Then a representative bank faces the following constraints at *t*:

$$b_t+m_t+s_t\le\omega-\tau+\ell_t, \tag{36}$$

$$\pi d_t^m(\omega-\tau)\le m_t\left(\frac{p_t}{p_{t+1}}\right), \tag{37}$$

$$(1-\pi)d_t^n(\omega-\tau)\le x(b_t+s_t)-I_t^c\left(\frac{p_t}{p_{t+1}}\right)\ell_t, \tag{38}$$

$$\ell_t\le\ell_t^c. \tag{39}$$

As before, the bank chooses values for reserves (m_t), interest-earning assets ($s_t + b_t$), and discount-window loans (ℓ_t), along with a return vector $\left(d_t^m, d_t^n\right)$, to maximize

$$\pi\frac{\left[d_t^m(\omega-\tau)\right]^{1-\rho}}{1-\rho}+(1-\pi)\frac{\left[d_t^n(\omega-\tau)\right]^{1-\rho}}{1-\rho}$$

subject to (36) through (39). With the reserve-deposit ratio γ_t defined as above, it is straightforward to show that the optimal reserve-deposit ratio is given by

$$\gamma_t=\frac{1+\left(\frac{\ell_t^c}{\omega-\tau}\right)\left(\frac{I_t-I_t^c}{I_t}\right)}{1+\left(\frac{1-\pi}{\pi}\right)I_t^{\frac{1-\rho}{\rho}}}\equiv\gamma(I_t)\left\{1+\left(\frac{\ell_t^c}{\omega-\tau}\right)\left(\frac{I_t-I_t^c}{I_t}\right)\right\}. \tag{40}$$

Thus, if the central bank sets $I_t^c = I_t$, then the optimal reserve-deposit ratio is exactly as before.

However, if it offers discount-window credit at a subsidy rate (i.e., if $I_t^c < I_t$), then banks' demand for reserves is enhanced, other things equal.

5.3 General Equilibrium with a Discount Window

In a competitive, perfect-foresight equilibrium, the demand for and supply of real balances must be equal. Thus,

$$m_t = (\omega-\tau)\gamma(I_t)\left\{1+\left(\frac{\ell_t^c}{\omega-\tau}\right)\left(\frac{I_t - I_t^c}{I_t}\right)\right\}, \quad t \geq 1. \tag{41}$$

In addition, the government budget constraint must be satisfied. This can be written as

$$\begin{aligned} g-\tau &= b_t - xb_{t-1} + xb_{t-1}^c\left(\frac{I_{t-1}-1}{I_{t-1}}\right) + \left(I_{t-1}^c - 1\right)\frac{p_{t-1}}{p_t}\ell_t^c \\ &= b_t - xb_{t-1} + \left(x - \frac{p_{t-1}}{p_t}\right)\left(b_{t-1}^c + \ell_{t-1}^c\right) + \left(I_{t-1}^c\frac{p_{t-1}}{p_t} - x\right)\ell_{t-1}^c \\ &= b_t - xb_{t-1} + xm_{t-1}\left(\frac{I_{t-1}-1}{I_{t-1}}\right) + x\ell_{t-1}^c\left(\frac{I_{t-1}^c - I_{t-1}}{I_{t-1}}\right), \quad t \geq 1, \end{aligned} \tag{42}$$

where the second equality uses the central bank's balance-sheet identity (33). Finally, $m_t \leq b_t^c$ need no longer hold. Hence, there is no analog to the equilibrium condition $(\omega-\tau)\gamma(I_t) \leq \bar{b}$ of Section 3. Thus, the use of the discount window as an instrument of monetary policy relaxes a potentially binding economic constraint.

Substituting (41) into (42) and rearranging terms yields the single equilibrium condition

$$\begin{aligned} g-\tau &= b_t - xb_{t-1} + x(\omega-\tau)\gamma(I_{t-1})\left(\frac{I_{t-1}-1}{I_{t-1}}\right)\left\{1+\left(\frac{\ell_{t-1}^c}{\omega-\tau}\right)\left(\frac{I_{t-1}-I_{t-1}^c}{I_{t-1}}\right)\right\} - x\ell_{t-1}^c\left(\frac{I_{t-1}-I_{t-1}^c}{I_{t-1}}\right) \\ &= b_t - xb_{t-1} + x(\omega-\tau)H(I_{t-1}) + x\ell_{t-1}^c\left(\frac{I_{t-1}-I_{t-1}^c}{I_{t-1}}\right)\left[H(I_{t-1})-1\right], \quad t \geq 1, \end{aligned} \tag{43}$$

where, as before, $H(I) = \gamma(I)(I-1)/I$. In addition, the treasury is assumed to control the level of real debt outstanding exogenously, allowing it to evolve according to (10).

From this analysis it is apparent that if the central bank charges market rates of interest on discount-window loans (that is, if $I_{t-1}^c = I_{t-1}$ for all *t*), then (43) is identical to (14). However, as already noted, the previous equilibrium condition (20) is no longer relevant. Thus, when banks pay the market rate of interest at the discount window, it is as if the economy of Sections 2 and 3 is always in case 1, independent of the total stock of outstanding debt.[11]

If $I_{t-1}^{c} < I_{t-1}$, then discount-window policy clearly affects the set of equilibria. The next subsection analyzes the steady states that arise under this situation. Considerations introduced by dynamical equilibria are very similar to those described in Section 3 and thus are not studied separately here.

5.3.1 Steady States

If the central bank sets the discount rate at $I^{c} = \eta I$, $\eta \in (0,1]$, η exogenously given, and sets the per capita volume of discount-window lending at a constant level of ℓ^{c}, then in a steady state (43) takes the form

$$\begin{aligned} \bar{b} &= \left(\frac{1}{x-1}\right)\left\{\tau - g + x(\omega-\tau)H(I) + x\ell^{c}(1-\eta)\left[H(I)-1\right]\right\} \\ &= \left(\frac{1}{x-1}\right)\left\{\tau - g - x\ell^{c}(1-\eta) + x\left[\omega-\tau+\ell^{c}(1-\eta)\right]H(I)\right\} \qquad (44) \\ &\equiv D(I;\eta). \end{aligned}$$

Equation (44) coincides with (16) if $\eta = 1$.

In contrast, when $\eta < 1$, the function $D(I;\eta)$ is as depicted in Figure 9 below. As η rises, the right side of (44) increases at each value of *I*. It follows that an increase in η reduces (increases) the steady-state value of the nominal rate of interest on the good (bad) side of the Laffer curve. Thus, the more heavily the central bank subsidizes discount-window lending, the higher (lower) are the nominal interest rate and the inflation rate on the good (bad) side of the Laffer curve. In particular, on the good side of the Laffer curve, subsidizing discount-window loans augments the demand for reserves by banks. However, this effect is not sufficient to offset the additional need for seigniorage revenue associated with subsidized—and therefore costly—discount-window lending.

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[11] However, if $\bar{b}$ satisfies the conditions of Proposition 1(b), it will still be the case that no equilibrium exists, regardless of whether the discount window is used as an instrument of policy. Discount-window use allows the central bank only to overcome problems associated with an inability to provide liquidity through open market operations.

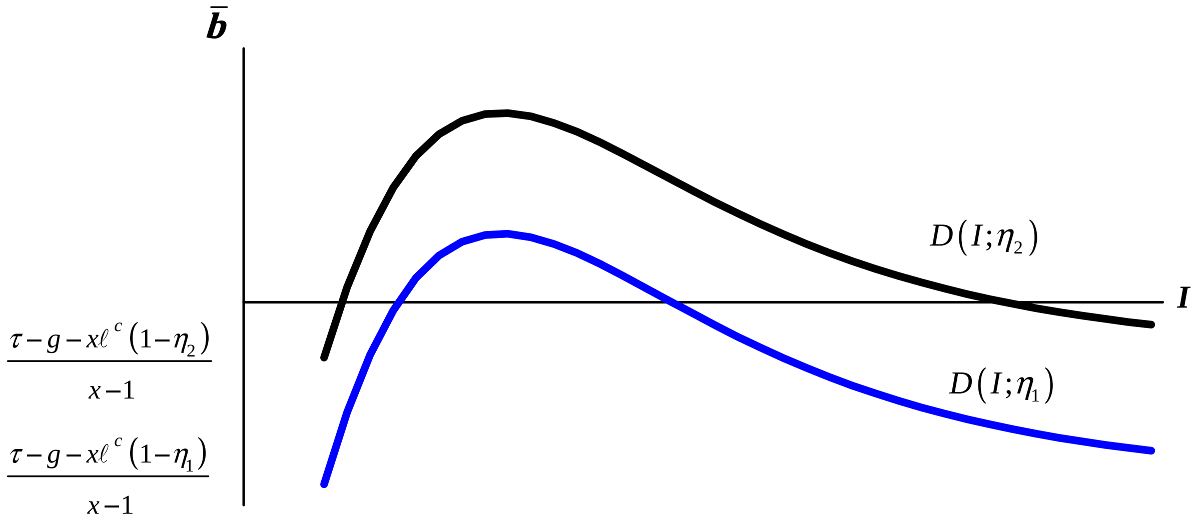

Figure 9—The Consequences of Changing the Discount Factor, $\eta_2 > \eta_1$

5.3.2 Steady-state Welfare

The final issue to consider regarding the discount window is whether there is a case for subsidizing discount-window lending (i.e., for setting $\eta < 1$). This subsection evaluates how steady-state welfare depends on η. It finds that if the economy is in a steady-state equilibrium on the bad side of the Laffer curve, then steady-state welfare is increased by subsidizing discount-window lending. The nature of the relationship between steady-state welfare and η is less straightforward on the good side of the Laffer curve. There, however, sufficiently small subsidies on discount-window loans can be shown to reduce steady-state welfare. Thus, the case for charging below-market rates of interest at the discount window, if monetary policy is going to be conducted via discount-window lending, depends on arguing that the economy is in an equilibrium on the bad side of the Laffer curve.

These results can be obtained by defining

$$\tilde{\gamma}(I) = \gamma(I)\left[1 + \left(\frac{\ell^c}{\omega - \tau}\right)\left(\frac{I - I^c}{I}\right)\right] = \gamma(I)\left[1 + (1-\eta)\left(\frac{\ell^c}{\omega - \tau}\right)\right], \tag{45}$$

where the second equality uses $I^c = \eta I$. It is easy to show that

$$d^m = \frac{x\tilde{\gamma}(I)}{\pi I},$$

$$d^n = \frac{x\left[1-\tilde{\gamma}(I)\right] + x(1-\eta)\left(\frac{\ell^c}{\omega-\tau}\right)}{1-\pi},$$

and that maximized depositor expected utility in a steady state, as a function of *I* and η, is given by the function $V(I,\eta)$, where *V* is defined by

$$(1-\rho)V(I,\eta) = \pi\left[\frac{x\tilde{\gamma}(I)}{\pi I}\right]^{1-\rho} + (1-\pi)\left\{\frac{x\left[1-\tilde{\gamma}(I)\right] + x(1-\eta)\left(\frac{\ell^c}{\omega-\tau}\right)}{1-\pi}\right\}^{1-\rho}.$$

Clearly,

$$V_1(I,\eta) = -\left(\frac{\pi}{I}\right)\left[\frac{x\tilde{\gamma}(I)}{\pi I}\right]^{1-\rho} < 0,$$

$$V_2(I,\eta) = -x\left(\frac{\ell^c}{\omega-\tau}\right)\left\{\frac{x\left[1-\tilde{\gamma}(I)\right] + x(1-\eta)\left(\frac{\ell^c}{\omega-\tau}\right)}{1-\pi}\right\}^{-\rho} < 0.$$

Since lower values of η are associated with lower values of *I* on the bad side of the Laffer curve, the following result is immediate.

Proposition 3. In steady-state equilibria on the bad side of the Laffer curve, subsidizing discount-window lending (i.e., setting $\eta < 1$) raises steady-state welfare.

The opposite result can be obtained regarding the good side of the Laffer curve by defining $\underline{I}(\eta)$ to be the smallest solution to (45) and $W(\eta)$ to be steady-state expected utility on the good side of the Laffer curve (that is, $W(\eta) \equiv V\left[\underline{I}(\eta),\eta\right]$). The following result then obtains; its proof appears in the appendix.

Proposition 4. On the good side of the Laffer curve, steady-state welfare is higher when discount-window loans are made at the market rate of interest than when they are slightly subsidized (that is, $W'(1) > 0$ holds).

Intuitively, if other things remain equal, depositors would benefit from slightly subsidized discount-window lending (choices of η close to, but less than, one). However, this benefit is more than offset by the higher nominal interest rate and the higher rate of inflation required to finance the subsidy. Thus, the policy of charging below-market interest rates at the discount window is beneficial only in equilibria on the wrong side of the Laffer curve.

Conclusion

For a central bank that conducts monetary policy through open market purchases and sales of government bonds, low levels of government debt outstanding can have major implications. As this paper has shown, if the value of the debt is too low, a lower bound is placed on the nominal rate of interest and thus the rate of inflation that can constitute equilibria. And, these lower bounds are large in the sense that they require nominal interest and inflation rates to lie on the Pareto inferior side of the seigniorage Laffer curve. Calculations using U.S. data suggest that debt levels in the U.S. could soon be low enough for these lower bounds to become binding, at least given current Federal Reserve operating procedures. If the government were to run a primary budget deficit, it could even be the case that excessively low levels of debt would interfere with the existence of an equilibrium.

Intuitively, low debt levels are problematic for the following reason. Low nominal rates of interest lead agents to demand high levels of liquidity. When debt levels are too low, central bank operating procedures prevent the supply of liquidity from being adequate to satisfy demand. As a result, low nominal rates of interest cannot be observed in equilibrium.

Matters become even worse when there are exogenously imposed limits on the amount of government debt that the central bank can hold. Currently, the Federal Reserve System has self-imposed limits on the fraction of the debt of various maturities that it can hold. These kinds of restrictions raise the debt level required to prevent lower bounds on nominal interest rates from becoming binding.

Central banks can avoid the implications of low debt levels by conducting monetary policy through a discount window. In theory, if the central bank charges market rates of interest on discount window loans, it can completely undo any limitations on its ability to provide liquidity that might be implied by low levels of government debt. In practice, however, at least in the U.S. today, discount-window borrowing is extremely low and would have to be stimulated for the discount window to be a viable means of conducting policy. The analysis here suggests that subsidizing discount-window loans will reduce steady-state welfare, at least in equilibria on the good side of the Laffer curve. A case for subsidizing discount-window lending would rely on arguing that the economy is likely to end up in equilibria on the wrong side of the Laffer curve.

Of course, the model presented in this paper abstracts from a number of factors that are likely to be relevant to the issues discussed. At a very basic level, it abstracts from real economic growth, which would tend to raise the demand for liquidity over time. Thus, allowing for growth would simply imply that the U.S. economy is likely to reach case 2 even sooner than suggested here.

The model also abstracts from other factors that are likely to affect the demand for liquidity, and hence $\bar{b}_1$. One is developments abroad, such as dollarization, that increase the demand for U.S. base money. Another factor is reductions in the use of cash in transactions, which also reduces the demand for base money (as, for example, in Schreft and Smith, 2000). Any tendency for the demand for base money to rise or fall over time would affect the estimate of $\bar{b}_1$.

The analysis of limits on how much debt the central bank can hold also ignores two important points. First, it assumes that debt markets remain competitive no matter how much of the debt is held by the central bank. Clearly, the case for imposing limits on a central bank's holdings of debt has greater support if government debt markets are thinner when the private sector holds a small fraction of the debt outstanding. Second, the model assumes that a second risk-free asset is available for agents to hold in addition to government bonds and that no risky assets are available. Thus, the model cannot assess the possible loss to the private sector if government debt disappears and so can no longer serve as a benchmark in pricing risky assets.

Finally, the analysis of the discount window abstracts from a number of issues as well. For example, it does not consider risk or collateral requirements associated with discount-window lending. A shortage of eligible collateral, for instance, could prevent the use of discount-window lending from completely undoing the consequences of low debt levels. The analysis also abstracts from the possibility that large volumes of discount-window lending would create moral hazard problems in banking. Clearly, these are considerations that could prevent discount-window lending from being a perfect substitute for open-market operations.

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Appendix

Proof of Lemma 2. That $H(1)=0$ is obvious. Part (b) of the lemma follows from the observation that $\lim_{I\to\infty}\gamma(I)=0$. Part (c) follows from differentiating the definition of $H(I)$ and using part (b) of Lemma 1.

For part (d), note that $H'(I)\geq 0$ holds iff

$$1\geq\left(\frac{1-\rho}{\rho}\right)(I-1)\left[1-\gamma(I)\right]\equiv Q(I) \tag{A.1}$$

Clearly $Q(1)=0$, $\lim_{I\to\infty}Q(I)=\infty$, and $Q'(I)>0$ for all $I > 1$. Thus, there is a unique value $\hat{I}>1$ satisfying (21). And, $H'(I)\geq 0$ holds iff $I\leq\hat{I}$, as claimed.■

Proof of Proposition 2. (a)(i). For any candidate steady state to be in the region of case 1, independent of the value of $\bar{b}$, (16) must be above (20) everywhere. This is the case if

$$\begin{aligned}(\omega-\tau)\gamma(I)&<\left(\frac{1}{x-1}\right)\left[\tau-g+x(\omega-\tau)H(I)\right]\\ &=\left(\frac{1}{x-1}\right)\left[\tau-g+x(\omega-\tau)\gamma(I)\left(\frac{I-1}{I}\right)\right]\end{aligned} \tag{A.2}$$

holds for all $I\geq 1$. It is easy to show that (A.2) is equivalent to

$$\left(\frac{x-I}{I}\right)\gamma(I)(\omega-\tau)<\tau-g\,. \tag{A.3}$$

Since the left side of (A.3) is decreasing in I for all $I\leq x$, (A.3) holds for all $I\geq 1$ if $(x-1)\gamma(1)(\omega-\tau)<\tau-g$. But this condition is equivalent to $(\omega-\tau)\gamma(1)<\left(1/(x-1)\right)\left[\tau-g+x(\omega-\tau)H(1)\right]$.

(a)(ii). The condition $(\omega-\tau)\gamma(1)>\dfrac{1}{x-1}\left[\tau-g+x(\omega-\tau)H(1)\right]$ implies that (16) is below (20) at $I = 1$. From Lemmas 1 and 2, (16) is increasing in I for $I<\hat{I}$, while (20) is decreasing. The condition

$$(\omega-\tau)\gamma(\hat{I})<\frac{1}{x-1}\left[\tau-g+x(\omega-\tau)H(\hat{I})\right] \tag{A.4}$$

guarantees that they intersect at some $I<\hat{I}$. Therefore (20) has at least one solution, and thus a value of $\bar{b}_1$ exists. It follows by definition that case 1 obtains for all $\bar{b}\geq\bar{b}_1$.

For case 2 to obtain for all $\bar{b} < \bar{b}_1$, it must be the case that

$$(\omega-\tau)\gamma(I) < \left(\frac{1}{x-1}\right)\left[\tau-g+x(\omega-\tau)H(I)\right] \tag{A.5}$$

for all $I \geq \hat{I}$. This can be established by noting that (A.5) is exactly (A.2), and thus equivalent to (A.3) for all $I \geq \hat{I}$. Condition (A.3) holds since its left side is decreasing in I for $I \in \left[\hat{I}, x\right]$, since the left side of (A.3) is negative for all $I > x$, and since, by assumption, (A.4) holds. In particular, (A.4) is equivalent to

$$\left(\frac{x-\hat{I}}{\hat{I}}\right)\gamma\left(\hat{I}\right)(\omega-\tau) < \tau - g .$$

It is now apparent that (24) has only one solution. Thus case 2 obtains for all $\bar{b} < \bar{b}_1$.

(b). Equations (16) and (20) intersect at values of I satisfying

$$(\omega-\tau)\gamma(I) = \left(\frac{1}{x-1}\right)\left[\tau-g+x(\omega-\tau)H(I)\right]. \tag{A.6}$$

(A.6) has the equivalent representation

$$\frac{\tau-g}{\omega-\tau} = \left(\frac{x-I}{I}\right)\gamma(I) \equiv F(I). \tag{A.7}$$

Clearly, $F(x)=0$ and $\lim_{I\to\infty} F(I)=0$. Moreover, $F'(I) \leq (\geq) 0$ for all $I \leq (\geq) \tilde{I}$, where $\tilde{I}$ is uniquely defined by $x \equiv \left[(1-\rho)/\rho\right]\left(\tilde{I}-x\right)\left[1-\gamma\left(\tilde{I}\right)\right]$. Thus, since $\tau - g < 0$, (A.7) has two (no) solutions if $F\left(\tilde{I}\right) < (>) \left[(\tau-g)/(\omega-\tau)\right]$. Rearranging terms gives the condition stated in the proposition.■

Proof of Proposition 4. Differentiating (44) with respect to η yields

$$\underline{I}'(\eta) = \frac{\ell^c\left\{H\left[\underline{I}(\eta)\right]-1\right\}}{\left[\omega-\tau+\ell^c(1-\eta)\right]H'\left[\underline{I}(\eta)\right]}. \tag{A.8}$$

Since $H(I) < 1$ for all *I*, and since $H'\left[\underline{I}(\eta)\right] > 0$, $\underline{I}'(\eta) < 0$.

It is easy to show that

$$W'(\eta) = V_1\left[\underline{I}(\eta),\eta\right]\underline{I}'(\eta) + V_2\left[\underline{I}(\eta),\eta\right]$$

$$= -\left(\frac{\pi}{I}\right)\left[\frac{x\tilde{\gamma}(I)}{\pi I}\right]^{1-\rho} \frac{\left(\frac{\ell^c}{\omega-\tau}\right)\left[H(I)-1\right]}{\left[1+(1-\eta)\left(\frac{\ell^c}{\omega-\tau}\right)\right]H'\left(\underline{I}(\eta)\right)} \tag{A.9}$$

$$-x\left(\frac{\ell^c}{\omega-\tau}\right)\left\{\frac{x\left[1-\tilde{\gamma}(I)\right]+x(1-\eta)\left(\frac{\ell^c}{\omega-\tau}\right)}{1-\pi}\right\}^{-\rho}.$$

Moreover, the first-order condition for the choice of the reserve-deposit ratio is

$$\left[\frac{x\tilde{\gamma}(I)}{\pi I}\right]^{-\rho} = I\left\{\frac{x\left[1-\tilde{\gamma}(I)\right]+x(1-\eta)\left(\frac{\ell^c}{\omega-\tau}\right)}{1-\pi}\right\}^{-\rho}. \tag{A.10}$$

Substituting (A.10) into (A.9) yields

$$W'(\eta) = -\frac{x\left(\frac{\ell^c}{\omega-\tau}\right)}{I}\left[\frac{x\tilde{\gamma}(I)}{\pi I}\right]^{-\rho}\left\{1+\left[\frac{\tilde{\gamma}(I)}{I}\right]\frac{\left[H(I)-1\right]}{\left[1+(1-\eta)\left(\frac{\ell^c}{\omega-\tau}\right)\right]H'\left(\underline{I}(\eta)\right)}\right\}. \tag{A.11}$$

Thus,

$$W'(1) = -\frac{x\left(\frac{\ell^c}{\omega-\tau}\right)}{I}\left[\frac{x\tilde{\gamma}(I)}{\pi I}\right]^{-\rho}\left\{1+\left[\frac{\tilde{\gamma}(I)}{I}\right]\frac{\left[H(I)-1\right]}{H'\left(\underline{I}(1)\right)}\right\}.$$

To prove the proposition, it remains to show that

$$1+\left\{\frac{\tilde{\gamma}\left[\underline{I}(1)\right]}{\underline{I}(1)}\right\}\left\{\frac{H\left[\underline{I}(1)\right]-1}{H'\left[\underline{I}(1)\right]}\right\}<0. \tag{A.12}$$

Condition (A.12) is equivalent to

$$1<\left[\frac{\tilde{\gamma}(I)}{H(I)}\right]\left\{\frac{H\left[\underline{I}(1)\right]}{\underline{I}(1)H'\left[\underline{I}(1)\right]}\right\}\left\{1-H\left[\underline{I}(1)\right]\right\}. \tag{A.13}$$

Using the definition of *H*, part (c) of Lemma 2, and

$$1-H(I) = \left\{\frac{I\left[1-\gamma(I)\right]+\gamma(I)}{I}\right\}$$

in (A.13) yields the equivalent condition

$$1<\frac{I\left[1-\gamma(I)\right]+\gamma(I)}{1-\left(\frac{1-\rho}{\rho}\right)\left[1-\gamma(I)\right](I-1)}. \qquad \text{(A.14)}$$

Since $1>\left[(1-\rho)/\rho\right]\{1-\gamma[\underline{I}]\}[\underline{I}-1]$ (that is, since $H'>0$ on the good side of the Laffer curve), (A.14) is clearly satisfied.■