ISSN 1936-5330

kcFED Research Working Papers

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Yulei Luo, Jun Nie, Xiaowen Wang, and Eric Young

July 2021
RWP 21-05
http://doi.org/10.18651/RWP2021-05

Federal Reserve Bank of Kansas City

Production and Inventory Dynamics Under Ambiguity Aversion*

Yulei Luo†
The University of Hong Kong

Jun Nie‡
Federal Reserve Bank of Kansas City

Xiaowen Wang§
Shanghai University of Finance and Economics

Eric R. Young¶
University of Virginia and Federal Reserve Bank of Cleveland

August 8, 2021

Abstract

We propose a production-cost smoothing model with Knightian uncertainty and ambiguity aversion to study the joint behavior of production, inventories, and sales. Our model can explain four facts that previous studies find difficult to account for simultaneously: (i) the high volatility of production relative to sales, (ii) the low ratio of inventory-investment volatility to sales volatility, (iii) the positive correlation between sales and inventories, and (iv) the negative correlation between the inventory-to- sales ratio and sales. We find that the stock-out avoidance

*We are grateful to Evan Anderson, Ken Kasa, Aubhik Khan, Camilo Morales-Jimenez, Jenny Xu, Chi-Wa Yuen, Hongsong Zhang, Lei Zhang, and seminar and conference participants at the University of Hong Kong, Shanghai University of Finance and Economics, 2021 Spring Fed Brownbag Seminar, and Simon Fraser University, the Midwest Macroeconomic Meeting, the Econometric Society China Meeting, and the 23rd and 27th Annual Conference on Computing in Economics and Finance (CEF) for helpful discussions and comments. We also thank Chenchuan Shi for excellent research assistance throughout this project. Luo thanks the General Research Fund (GRF No. HKU749711 and HKU17500619) in Hong Kong for financial support. Wang acknowledges the financial support from the National Science Foundation of China (No. 71850002). The views expressed here are the opinions of the authors only and do not necessarily represent those of the Federal Reserve Banks of Cleveland or Kansas City or the Federal Reserve System.

†Faculty of Business and Economics, The University of Hong Kong, Hong Kong. E-mail: yulei.luo@gmail.com.

‡Research Department, Federal Reserve Bank of Kansas City. E-mail: jun.nie@kc.frb.org.

§Institute of Advanced Research and Key Laboratory of Mathematical Economics, Shanghai University of Finance and Economics, Shanghai, China. E-mail: wang.xiaowen@mail.shufe.edu.cn.

¶Department of Economics, University of Virginia, Charlottesville, VA, and Research Department, Federal Reserve Bank of Cleveland, Cleveland OH. E-mail: ey2d@virginia.edu.

motive (Kahn 1987) emerges endogenously in our model, reconciling the long debate in the inventory literature over the production- cost smoothing and the stock-out avoidance models.

Keywords: Ambiguity Aversion, Robustness, Knightian Uncertainty, Inventories, Production-Cost Smoothing.

JEL Classification Numbers: D83, E21, F41, G15.

1 Introduction

In more than half a century, economists have made great strides toward better understanding firms' inventory decisions. This progress is unsurprising given the disproportionate role inventory investment plays in explaining fluctuations in output.[1] Most previous studies on inventory dynamics are based on the linear-quadratic framework developed by Holt et at. (1960), which assumes firms' production and inventory decisions are associated with convex costs and stochastic demand. Three prominent theories have emerged from this framework: the production smoothing (PS) model, which assumes firms use inventories to minimize production fluctuations given stochastic demand; the production cost smoothing (PCS) model, which introduces shocks to the marginal cost of production and assumes that firms hold inventories to smooth the production costs rather than production levels; and the stock-out avoidance (SOA) model, which assumes firms have incentive to accumulate inventories because they cannot short sell their products without cost.

However, researchers have questioned whether these theories can rationalize two important aspects of the data: the higher volatility of production relative to sales and the procyclicality of inventory investment. The consensus in the literature is that the PS model fails to generate a volatility of output higher than that of sales (which is well known as the "production smoothing puzzle").[2] Although the PCS model can generate a higher output volatility with the help of large cost shocks to production, it fails to generate procyclical inventory investment. Though the SOA model appears to be more consistent with the data in these two dimensions, the existing modeling strategies are a little ad hoc and lack micro foundations.[3]

One common feature of these model is that they assume firms know exactly the functional forms of the true demand and cost-shock processes. For example, a typical assumption used in the literature is AR(1) specifications for the demand process (Blinder 1982) and cost shock process (Eichenbaum 1989). So firms consider them as true data-generating processes for demand and cost shocks, and use them to form expectations for next period's demand and cost shock. In practice, however, the true demand and cost-shock processes may either be difficult to precisely measure or completely unobservable. Thus, it is plausible that firms' reference models of these processes are

[1] As emphasized in the literature, inventory investment constitutes less than 1 percent of real GDP in most countries, but accounts for a significant fraction of real GDP fluctuations. For example, in the United States, the decline in inventory investment in the two most recent severe recessions, 1990-1991 and 2007-2009, explains 49 percent and 42 percent of the total decline in output, respectively.

[2] The production smoothing puzzle was first raised by Blanchard (1983). If inventories are used to absorb fluctuations in demand, the model predicts that production will be smoother than sales. However, the data show the opposite. See Blinder (1986), Blinder and Maccini (1991), and Lai (1991) for detailed discussions on this puzzle.

[3] Section 5.3 provides more detailed discussions on the two different ways to model the SOA.

misspecified in some way and it is plausible, too, that firms may be aware of this potential for error and alter their investment decisions accordingly.

In this paper, we study the implications of firms' uncertainty about these demand and cost-shock processes for the joint dynamics of inventories, production, and sales. Specifically, we expand an otherwise standard PCS model by assuming i) that firms face Knightian uncertainty (or model uncertainty) about the true dynamics of sales and production costs; and ii) that aversion to this uncertainty leads them to make investment decisions under the worst-case scenario. We describe firms' aversion to ambiguity about the true dynamics of sales and production cost using the preference for robustness proposed by Hansen and Sargent (2007). In robustness models, agents have in mind a reference model that represents their best estimate of the model economy. However, because they are worried that this reference model may be incorrect in some hard-to-specify way, they make their optimal decisions under the worst-case scenario. We assume that firms are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they choose their decisions *as if* the subjective distribution over shocks was chosen by an evil agent in order to maximize their expected total costs.[4] That is, when making production and inventory decisions, firms will take into account all possible processes of sales and the cost shocks that are statistically indistinguishable to a certain degree (this degree is also called the degree of model uncertainty or Knightian uncertainty, and will be carefully defined later).

Our main result is that once we take Knightian uncertainty and ambiguity aversion into account, a standard PCS model can explain not only the two aspects of the data mentioned above but also other important dimensions. In addition, introducing ambiguity aversion into our PCS model allows the SOA motive to emerge endogenously, reconciling the PCS model with the SOA model, due to a precautionary savings motive. These findings suggest that embedding ambiguity aversion into this framework is a promising and perhaps necessary step in explaining the dynamics of inventory and production.

The key channel through which ambiguity aversion influences production and inventory dynamics is as follows. Firms with ambiguity aversion make production and inventory decisions under the worst-case scenario. In particular, these firms are more concerned about a high level of inventories and the resulting management costs than firms without ambiguity aversion and thus tend to produce less in response to a reduction in sales or an increase in production costs. Sim-

[4] In this paper, we follow Cagetti et al. (2002), Hansen and Sargent (2007, 2015), and Bidder and Smith (2012) in studying a model with ambiguity aversion described by "*multiplier*" preferences. For other specifications and applications of ambiguity aversion and robustness in macroeconomics, see Nishimura and Ozaki (2007), Ilut and Schneider (2014), Ilut et al. (2020), and others.

ilarly, ambiguity-averse firms tend to produce more in response to an increase of sales or drop in production costs, as they are more concerned about high production costs in the future.

The above key mechanism enables our model to explain four dimensions of data. First, as firms adjust production more aggressively in response to sales shocks, the relative volatility of production to sales increases, helping to resolve the production-smoothing puzzle. Second, when the response of production to sales is larger than one-to-one (say, due to a stronger degree of ambiguity aversion), the response of inventories can move in the same direction as sales, producing procyclical inventory investment. Third, as the response of inventory investment to sales shocks changes from negative to positive as ambiguity aversion rises, the volatility of inventory investment first declines before rising. Fourth, the inventory-to-sales ratio becomes countercyclical if ambiguity aversion is sufficiently strong, because a sufficiently strong degree of ambiguity aversion successfully prevents firm inventories from turning negative, an important point linked to the SOA motive.

Although our model allows model uncertainty, it can still be solved easily using standard tools, enabling us to use GMM to estimate the four key model parameters-the persistence and volatility of the cost shock, the relative importance of inventory holding cost, and the degree of ambiguity aversion (or the amount of model uncertainty). To discipline the four model parameters, we include four key moments in our estimation: (a) the relative volatility of production to sales; (b) the relative volatility of inventory investment to sales; (c) the correlation between inventory investment and sales; and (d) the correlation between the inventory-sales ratio and sales. We include (a) and (c) as they are two key aspects intensively discussed in the literature to examine different theories. We include (b) because the volatility of inventories dynamics has been highlighted in the literature explaining output volatility. And we include (d) because it is discussed in recent DSGE models studying business cycles (for example, see Wen, 2011; Kryvtsov and Midrigan, 2013). At the estimated parameter values, our model explains four moments reasonably well. The uncovered value of the model uncertainty parameter implies a 22% probability that the best estimated processes (which are AR(1) processes) cannot be statistically distinguished from the worst-case models based on a likelihood ratio test. This value is in the reasonable range of the values reported in the robust control literature and implies that the model's success is not due to unreasonable fears of model misspecification.

A natural question is whether introducing ambiguity aversion is just adding one free parameter to the model. To address this concern, we further conduct two sets of analyses. First, we examine how the four key moments vary with each of the key model parameters. We find that though quantitatively the four moments are sensitive to each of the parameters, the ambiguity aversion parameter drives the key changes that are consistent with the data. Second, we solve and estimate a rational expectations (RE) version of the model without ambiguity aversion. We find that this RE model fails to explain either the procyclical inventory investment or the countercyclical inventory-to-sales ratio. Although the RE model can generate an output volatility higher than sales volatility, it requires the cost shock process to be substantially larger than that used in our benchmark model and, quantitatively, it still cannot match the data, confirming the insights from previous work (Blinder 1986; Eichenbaum 1989).

As mentioned earlier, we show that the presence of Knightian uncertainty provides an endogenous mechanism for generating a stock-out avoidance motive. The stock-out avoidance motive was first proposed by Kahn (1987) and has been widely used in the inventory literature, as it successfully resolves the production-smoothing puzzle. Existing models incorporate the stock-out avoidance motive in two ways. The first is to directly impose a non-negative constraint on inventories (Kahn 1987; Wen 2005) so firms have to accumulate more inventories to avoid possible stock-outs. The second is to include an "accelerator" term that represents a quadratic cost associated with allowing inventories to deviate from some fixed proportion of sales (Eichenbaum, 1989; Ramey, 1991; and Ramey and West, 1999). Notice that this cost term may not completely rule out negative inventories.[5] We show that our model with model uncertainty endogenously avoids negative inventories with a reasonable degree of ambiguity aversion. The intuition is that firms with a stronger degree of ambiguity aversion have a larger incentive to accumulate inventories as their decisions are based on the worst-case scenario, which generates a form of precautionary savings. Therefore, taking Knightian uncertainty into account can endogenously generate the stock-out avoidance motive even though the interpretation is completely different.

We choose the linear-quadratic Gaussian (LQG) framework to study the implications of ambiguity aversion on the dynamics of inventory and production mainly because this is the primary framework used in this literature. This makes it easier for us to contrast our results with the existing results from other models based on the same framework. In addition, as shown in Hansen and Sargent (2007, 2010), within the LQG setting using relative entropy to measure model misspecification leads to a simple generalization to the ordinary LQG dynamic program problem, which keeps our model tractable and allows us to fully inspect the mechanism through which ambiguity aversion affects the dynamics of production and inventories. There are also a few recent papers that study inventory dynamics and business cycles based on general equilibrium models (Khan and Thomas, 2007; Wen, 2011; Wang et al., 2014). Though these DSGE models can include more elements and be used to study a wide range of other important issues on business cycles, the complexities in these models may muddy the intuition we want to highlight. We want to use our analysis to make a clear point that once we allow for a moderate degree of ambiguity aversion, a standard linear-

[5] In one simulation exercise, we show it requires the parameter on the targeted proportion of sales to be significantly higher than the empirically plausible level to guarantee all inventories are positive.

quadratic inventory model can explain the data reasonably well, even without explicitly assuming the stockout avoidance motive.

Literature Review Our paper is related to two branches of literature. First, it is related to the long branch of literature examining the joint dynamics of production, inventories, and sales based on the linear-quadratic framework dating back to Holt et al. (1960). This literature in general studies different theories' implications for inventory and production dynamics focusing on two aspects. Blanchard (1983) is the first to show the production-smoothing model cannot explain why output volatility is higher than sales volatility in the data. To increase the relative volatility of output to sales, Blinder (1986), West (1987), and Eichenbaum (1989) convert the production smoothing model into a production-cost smoothing model by introducing shocks to technology and production cost. Eichenbaum (1989) provides empirical evidence for the PCS model over the PS model. Kahn (1987) and Maccini and Zabel (1996) show that incorporating the SOA motive can also help explain the relative high output volatility. Wen (2005) shows that the PCS model cannot explain the procyclical inventory investment, while the SOA model can. Our paper contributes to this literature by showing that incorporating ambiguity aversion not only enables the linear-quadratic inventory framework to explain the data well, but also to endogenously generate the SOA motive.

Second, our paper contributes to the branch of literature on Knightian uncertainty, ambiguity aversion, and robustness. Many recent papers have shown the usefulness of viewing agents as having (potentially) misspecified models and being aware of this fact. See, for example, Hansen (2007), Nishimura and Ozaki (2007), Bidder and Smith (2012), Luo et al. (2012), Djeutem and Kasa (2013), Ilut and Schneider (2014), Hansen and Sargent (2015), and Ilut et al. (2020). To the best of our knowledge, we are the first to introduce Knightian uncertainty and ambiguity aversion into the inventory literature.

The rest of paper is organized as follows. Section 2 presents the data and facts. Section 3 introduces our benchmark model of robust production and inventory decisions and explains how to solve the model numerically. Section 4 delivers the model's quantitative implications and show how the model fits the data well along the key aspects. Section 5 provides further discussions. Section 6 concludes.

2 Data and Facts

In this section, we document some facts regarding the joint behavior of production, inventories, and sales. We focus primarily on the manufacturing sector, where inventories play an important role in the production process. As a robustness check, we also report results using data from the wholesale and retail sectors.

Our monthly data comes from the Bureau of Economic Analysis (BEA) and covers the 1967m1–2016m12 period. To exclude the volatile period around the Great Financial Crisis (GFC), we also report statistics based on a sample prior to the GFC, i.e., 1967m1–2007m9.[6] Following Blinder and Maccini (1991), we define the inventory stock as the sum of the final goods inventory plus work-in-progress (WIP) inventory. We include WIP following the arguments in Ramey and West (1999) that changes in final goods inventory only accounts for a small and relatively-smooth fraction of inventory investment, and therefore does not present a full picture.

Following Mosser (1991), Blinder (1986), and Maccini and Pagan (2013), we define our production measure as the sum of sales and inventory investment. We take logarithms of the raw data and detrend using the HP filter.[7] Sales, x_t, are from different sectors, and we take the logarithm and detrend it to get HP filtered $\log(x_t)$. Table 2 gives the estimated persistence of sales using the whole sample as well as the before GFC sample. Monthly output are constructed from $y_t = x_t + \Delta i_t$, where i_t is the sectorial inventory stock. For wholesale and retail sectors, y_t is associated with deliveries. We take logarithm of y_t and detrend using HP filter. Since inventory investment can be negative, the measurement of Δi_t is different. We define $\Delta i_t = (i_t - i_{t-1})/x_t$. The other exception is the inventory/sales ratio, i_t/x_t, which is defined as level of inventory stock divided by level of sales. The data moments corresponding to $\mu_{yx} \equiv \operatorname{var}(y_t)/\operatorname{var}(x_t)$, $\mu_{ix} \equiv \operatorname{var}(\Delta i_t)/\operatorname{var}(x_t)$, $\rho_{ix} \equiv \operatorname{cov}(\Delta i_t, x_t)/\left(\sqrt{\operatorname{var}(\Delta i_t)}\sqrt{\operatorname{var}(x_t)}\right)$ and $\rho_{i/x,x} \equiv \operatorname{cov}(i_t/x_t, x_t)/\left(\sqrt{\operatorname{var}(i_t/x_t)}\sqrt{\operatorname{var}(x_t)}\right)$ are then calculated based on the treated data accordingly. The persistence of sales are estimated from detrended $\log(x_t)$.

Tables 1 summarizes the key moments we focus on with GMM Newey-West standard errors in parentheses. Panel A reports the statistics for the manufacturing sector, while Panels B and C report results for the wholesale and retail sectors. We will mainly focus on Panel A when explaining the results, but the basic pattern is similar across all sectors. As the first row of Panel A shows, the ratio of production volatility to sales volatility is larger than 1. As the middle and bottom panels show, this finding is robust to including the wholesale and retail sectors.[8] The statistics are

[6]In BEA, the 1967 – 1996 data are measured as chained 1996 dollars to the SIC level, and data in 1997 and onwards are measured as chained 2009 dollar to the NAICS level. After adjusting for deflation and possible small jumps because of different estimation standards, we connect the two sub-sample periods.

[7]Blinder and Maccini (1991) detrend the data by regressing on a constant, a time trend, and an OPEC variable. They find that the OPEC variable has a very small effect on the results. In our data, the persistence of linearly-detrended sales is close to 1, suggesting a random walk process. Maccini et al. (2015) find that a random walk sales process can produce a conditional variance ratio greater than 1.

[8]For these two sectors, we define production as sales plus inventory investment in the same way. We follow the literature and call it "deliveries" rather than production.

all significant at the 1% level.

The second row of Panel A shows the relative volatility of inventory investment to sales. Although the ratio looks small, once one notes that inventory investment is less than 1 percent of sales, it looks much bigger. This ratio is substantially higher if we include either the wholesale or the retail sector.

The third row of Panel A displays the correlation between inventory investment and sales. Inventory investment is procyclical – its correlation with sales is 0.3 in the manufacturing sector.[9] Again, this positive correlation also shows up in the wholesale sector and the retail sector.

The fourth row of Panel A reports the correlation between the inventory-to-sales ratio and sales, which is negative. The counter-cyclical inventory-sales ratio is also documented in Khan and Thomas (2007, 2016), Wen (2011), and Kryvtsov and Midrigan (2013).

3 A Production-Cost Smoothing Model With Ambiguity Aversion

In this section, we will introduce our production-cost smoothing model with ambiguity aversion. We first describe the basic model environment and then introduce Knightian uncertainty and ambiguity aversion.

3.1 The Basic Model Environment

The basic model follows the literature on the production-cost smoothing model (Blinder 1986 and Eichenbaum 1989) closely.[10] Our ultimate goal is to examine how uncertainty about the sales process and the production cost interact with the preference for robustness and affects the joint dynamics of production, inventories, and sales. Specifically, following Blinder (1986) and Eichenbaum (1989), we assume that the production cost function of the representative firm is:

$$C(y_t) = \alpha_y \Gamma_t y_t + \frac{1}{2}\alpha_y y_t^2, \tag{1}$$

[9] Wen (2011) finds that inventory investment is countercyclical within a high frequency band (2-3 quarters) for quarterly data. We find with our monthly data inventory investment is also countercyclial within a high frequency band (0.7-2 quarters).

[10] Eichenbaum (1989) finds supportive empirical evidence for the production-cost smoothing model, in which inventories serve to smooth production cost rather than as a buffer stock to production levels. Eichenbaum (1984), Blinder (1986), and West (1987) retain the assumption of convex adjustment cost functions and considered production-cost smoothing models rather than the production-level smoothing models.

where y_t denotes the production level for the firm in period t. The condition, $\alpha_y > 0$, indicates that we assume the firm operates in a region of rising marginal costs. This function thus embodies the production *level* smoothing motive. Γ_t is a stochastic shock to the marginal cost of output, so firms have the incentive to raise production when this cost is low, provided the shock persists into the future. That is, (1) also embodies the production *cost* smoothing motive. The firm observes the cost shock before choosing the production plan.

The cost shock is governed by the following AR(1) process:

$$\Gamma_{t+1} = \rho_\Gamma \Gamma_t + w_{t+1}, \tag{2}$$

where $\rho_\Gamma \in [0, 1)$ is the persistence coefficient and w_t is an iid Gaussian innovation to the cost with mean 0 and variance Ψ.

The inventory holding cost function is:

$$H(i_t) = \frac{1}{2}\alpha_i i_t^2, \alpha_i > 0, \tag{3}$$

where i_t denotes the inventory level for the firm at the end of period t. The coefficients, α_i and α_y, govern the relative importance of the backlog cleanup motive and the production-smoothing motive in the firm's cost structure, respectively. We define $\alpha_{iy} \equiv \alpha_i/\alpha_y$ to represent this relative importance, as only this ratio plays a role in the model. In some studies, such as Kahn (1987), Eichenbaum (1989), Ramey (1991), and Ramey and West (1999), the inventory holding cost takes the form $H(i_t) = \alpha_i \left(i_{t-1} - \alpha_x x_t\right)/2$, where x_t is the amount of sales for the firm in t and $\alpha_x > 0$ governs the stockout avoidance motive, which induces the firm to hold inventories even without production cost considerations. To keep our baseline model as simple as possible, we focus only on the case $\alpha_x = 0$. In Section 5.3, we show that model uncertainty can provide a microfoundation for the stockout avoidance motive.

The accounting identity relating output, sales, and inventories is:

$$y_t = x_t + i_t - i_{t-1}, \tag{4}$$

where x_t is the amount of sales for the firm in t. Given sales and initial inventories, the firm makes optimal production and inventories to minimize the following discounted expected cost:

$$v(x_t, i_{t-1}) = \min_{\{i_t, y_t\}} \mathbb{E}_t \left\{ \sum_{j=t}^{\infty} \left(\frac{1}{R}\right)^{j-t} \left[C(y_t) + H(i_t)\right] \right\}, \tag{5}$$

subject to (1), (3), and (4).

To close the model, we need to specify the exogenous sales processes. The sales process is governed by the following AR(1) process:

$$x_{t+1} = \rho x_t + \varepsilon_{t+1}, \tag{6}$$

where $\rho \in [0, 1]$ is the persistence coefficient and ε_{t+1} is an iid Gaussian innovation to sales with mean $(1-\rho)\overline{x}$ and variance Ψ. Here, ε_t can be interpreted as the demand shock. In addition, we assume $\text{corr}(\varepsilon_t, w_t) = 0$, i.e., there is no correlation between the innovations to the cost and sales shocks.

Finally, the *timing* of our model economy is as follows. In period t, a firm currently holding an inventory stock i_{t-1} observes the demand and cost shocks, produces an amount of y_t of a single good, and sells an amount of x_t at the prevailing price. Production is instantaneous, and there is no delay between production and sales. In this sense, inventory is not a speculative tool to meet unexpected demand, but rather a buffer tool to smooth costly production. Finally, we assume that there are also no intermediate goods in our model and all goods are finished goods or works in progress and are free of physical depreciation.

3.2 Introducing Model Uncertainty Due to Robustness

In this section, we model firms' aversion to ambiguity about the true dynamics of sales and production cost using the preference for robustness (RB) proposed by Hansen and Sargent (2007). In the presence of RB, the inventory model proposed in Section 3.1 cannot be solved analytically, so we solve the model as a linear-quadratic regulator problem.

Following Hansen and Sargent (1995, 2007, 2010) and Luo and Young (2010), we use the risk-sensitivity operator, a special case of the recursive utility, to characterize the preference for robustness. The robust control version of the inventory model proposed in Section 3.1 can thus be written as:

$$v(i_{i-1}, x_t, \Gamma_t) = \min_{\{y_t\}} \left\{ \frac{1}{2}\alpha_y y_t^2 + \alpha_y \Gamma_t y_t + \frac{1}{2}\alpha_i i_t^2 + \beta \mathcal{R}_t v(i_t, x_{t+1}, \Gamma_{t+1}) \right\}, \tag{7}$$

subject to (2), (4), and the distorted expectation operator $\mathcal{R}$ is given by:

$$\mathcal{R}_t \left[v(i_t, x_{t+1}, \Gamma_{t+1}) \right] = -\frac{2}{\alpha} \log \mathbb{E}_t \left[\exp\left(-\frac{\alpha}{2} v\left(i_t, x_{t+1}, \Gamma_{t+1}\right) \right) \right],$$

where $\alpha > 0$ measures the degree of robustness in the model. The operator $\mathcal{R}$ distorts the usual conditional expectation with a single parameter $\alpha > 0$; α is also called the risk sensitivity parameter. We use the risk sensitivity representation because solving a robust control problem by imposing a constraint on conditional relative entropy is equivalent to an agent having a preference that applies an exponential transform to the continuation value. Replacing with the form of $\mathcal{R}$ delivers the risk-sensitive evaluation used in control theory, and the same results can be obtained as when solving a typical max-min problem with a penalty term defined in robust control theory by setting $\alpha = -\theta^{-1}$.[11] If $\alpha = 0$, the utility reduces to the usual rational expectations expected utility specification, which is equivalent to solving a max-min problem with θ approximating to $+\infty$.

Denote $s_t = \begin{bmatrix} i_{t-1} & x_t & \Gamma_t \end{bmatrix}^T$ as the state vector. Using (4), (7) can be rewritten as:

$$v\left(i_{t-1}, x_t, \Gamma_t\right) = \min_{\{y_t\}} \left\{ R y_t^2 + 2 s_t^T W y_t + s_t^T Q s_t + \beta \mathcal{R}_t \left[v\left(i_t, x_{t+1}, \Gamma_{t+1}\right)\right] \right\}, \tag{8}$$

subject to the state transition equations:

$$s_{t+1} = A s_t + B y_t + C \overrightarrow{\varepsilon}_{t+1}, \tag{9}$$

where $R = 1 + \frac{\alpha_y}{\alpha_i}$, $W = \begin{bmatrix} 1 \\ -1 \\ \frac{\alpha_y}{\alpha_i} \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho_\Gamma \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ \omega & 0 \\ 0 & \omega_\Gamma \end{bmatrix}$, and $\overrightarrow{\varepsilon}_{t+1} = \begin{bmatrix} \varepsilon_{t+1} \\ w_{t+1} \end{bmatrix}$.

To solve the linear quadratic robust control problem, we use the adapted ordinary optimal linear regulator proposed in Hansen and Sargent (2007). The first order conditions of the problem requires us to solve an algebraic Ricatti equation for P:

$$P = Q + \beta A' \mathcal{D}(P) A - (\beta A' \mathcal{D}(P) B + W)(R + \beta B' \mathcal{D}(P) P B)^{-1} (\beta B' \mathcal{D}(P) P A + W'), \tag{10}$$

where $\mathcal{D}(P) = P + \alpha P C \left(I - \alpha C' P C\right)^{-1} P C'$. Under some regularity conditions on the cost functions, the Ricatti equation has a unique positive semidefinite solution.

Since the model with ambiguity aversion implies that firms behave *as if* they hold worst-case beliefs, optimal outcomes can be identical to that obtained in the model with expected cost minimizers who are *endowed* with those worst-case beliefs. However, this observational equivalence does not imply that the two models are identical – worst-case beliefs would not generally be policy-invariant, and would therefore change with changes in the environment.

3.3 Robust Policy Rules and the Stochastic Properties of the Joint Dynamics of Production, Inventories, and Sales

The following proposition summarizes the solutions to the optimization problem described by (8)-(9):

[11] See Appendix 7.1 for a proof on the equivalence between the risk-sensitivity specification and the robust control specification proposed in Hansen and Sargent (2010).

Proposition 1 *Given α, the production and inventory policy functions can be written as:*

$$y_t^* = \mu_i i_{t-1} + \mu_x x_t - \mu_\Gamma \Gamma_t, \tag{11}$$

$$i_t^* = \lambda_i i_{t-1} + \lambda_x x_t - \lambda_\Gamma \Gamma_t, \tag{12}$$

where $\lambda_i = 1 + \mu_i$, $\lambda_x = -1 + \mu_x$, $\mu_\Gamma = \lambda_\Gamma$.

Proof. As mentioned in Section 3.2, solving the robust dynamic programming numerically yields the optimal control (11). Combining the accounting equation, $i_t - i_{t-1} = y_t - x_t$, with (11) yields (12). ■

We make several comments based on the solutions above to help understand how ambiguity aversion influences the joint dynamics of production, sales, and inventories. First, as a general comment, note that as $\alpha \to 0$, $\mathcal{R}_t$ becomes the ordinary conditional expectation operator $\mathbb{E}[\cdot|\mathcal{I}_t]$, and the coefficients on i_{t-1}, x_t, and Γ_t converge to the the solutions in (15) and (16) from a standard linear rational expectations problem:

$$i_t = \lambda_i i_{t-1} + \lambda_x x_t - \lambda_\Gamma \Gamma_t, \tag{13}$$

where $\lambda_i = \lambda_1$, $\lambda_x = -\lambda_1(R-\rho)/(R-\rho\lambda_1)$, and $\lambda_\Gamma = \lambda_1(R-\rho_\Gamma)/(R-\rho_\Gamma\lambda_1)$. The production policy is

$$y_t = \mu_i i_{t-1} + \mu_x x_t - \mu_\Gamma \Gamma_t, \tag{14}$$

where $\mu_i = \lambda_1 - 1 < 0, \mu_x = 1 + \lambda_x > 0$, $\mu_\Gamma = \lambda_\Gamma$, x_t is governed by (6), Γ_t is governed by (2), and λ_1 is the smaller root ($0 < \lambda_1 < 1$) satisfying $\lambda_1 + R/\lambda_1 = 1 + R + R\alpha_i/\alpha_y$.

Next, we focus on the two key coefficients, μ_x and λ_x, to see how ambiguity aversion (or robustness) changes the responses of production and inventory investment to sales.[12] As illustrated in the top right panel of Figure 2, as the degree of ambiguity aversion increases (α increases), the response of production to sales shocks increases; importantly, the response rise above 1. Production can increase more than the increase in sales, increasing production volatility and potentially leading to a positive correlation between inventory investment and sales.

Furthermore, as the top right panel of Figure 3 shows, as the degree of ambiguity aversion increases, the response of inventory investment to sales (λ_x) rises from negative to positive, which is consistent with the above insight that a larger α can lead to a positive correlation between inventory investment and sales. In addition, the fact that the absolute value of (λ_x) first declines

[12] The coefficient, λ_x, can be interpreted as the response of either inventories (i_t) or inventory investment ($i_t - i_{t-1}$) to sales.

and then increases suggests that the variance of inventory investment is likely to follow the same pattern. In other words, the volatility of inventory investment has a non-monotonic relationship with the degree of ambiguity aversion – the volatility first declines and then increases as the degree of ambiguity aversion rises. We will explore this issue more in the quantitative section.

The following proposition summarizes the model's predictions on the four targeted moments.

Proposition 2 *Under the RB and PCS assumptions, using (11) and (12), we can compute the following four targeted moments: (i) the relative volatility of production to sales* ($\mu_{yx} \equiv \text{var}(y_t)/\text{var}(x_t)$)*, (ii) the relative volatility of inventory investment to sales* ($\mu_{ix} \equiv \text{var}(\Delta i_t)/\text{var}(x_t)$)*, (iii) the contemporaneous correlation between the change in inventories with sales* ($\rho_{ix} \equiv \text{cov}(\Delta i_t, x_t)/\sqrt{\text{var}(\Delta i_t)}\sqrt{\text{var}(x_t)}$)*, and (iv) the contemporaneous correlation between the inventory-to-sales ratio and sales* ($\rho_{i/x,x} \equiv \text{corr}(i/x, x)$)*.*

Proof. See Appendix 7.2 for the detailed expressions for the first three targeted moments. Note that there is no explicit expression for moment (iv). ■

Given the complexity of the expressions, we cannot learn much from them. In the next section we will quantitatively evaluate the model's implications on explaining these four dimensions. In Section 5.1, we also derive some analytical solutions to the model without ambiguity aversion and robustness to help explain why the basic model fails.

4 Model's Quantitative Implications

In this section, we estimate the model parameters and quantitatively examine the model's ability to fit the data.

4.1 Estimating the Parameters

There are four key parameters we need to pin down in this model: the degree of ambiguity aversion (α), the persistence of cost shock (ρ_c), the volatility of the cost shock innovation(Ψ), and the relative importance of the inventory holding cost (α_{iy}). We jointly estimate these parameters to match: (1) the relative volatility of production to sales ($\text{var}(y)/\text{var}(x)$), (2) the relative volatility of inventory investment to sales ($\text{var}(\Delta i)/\text{var}(x)$), (3) the contemporaneous correlation between inventory investment and sales ($\text{corr}(\Delta i, x)$), and (4) the correlation between the inventory-sales ratio and sales ($\text{corr}(i/x, x)$). The estimation is based on a Generalized Method of Moments (GMM) approach (see Appendix 7.3 for details).

The estimated values of the parameters are reported in Table 3. Our baseline model uses the values from the GMM estimation with the optimal weighting matrix. As a robustness check, we also report the results using an identity weight matrix; as the table shows, the estimated values of the parameters are similar under these two approaches.[13] The estimated parameter values are also similar to the previous estimates in the literature. In particular, the estimate of the persistence coefficient of the cost shock is a bit lower than the values estimated in Eichenbaum (1989), while the estimate of the relative importance of inventory holdings is within the range summarized in Ramey and West (1999). To obtain the RE model-implied moments, the RB parameter is set to be zero with the same set of other parameters.

To provide a more meaningful interpretation for the ambiguity aversion and robustness parameter (α), we follow Anderson et al. (2003) and Hansen and Sargent (2007) by calculating the detection error probability (DEP) (see Appendix 7.4 for the details). The resulting DEP is 0.22, which means that there is 22% chance that a likelihood ratio test will improperly select between the approximating and the distorted models. This value is within the reasonable range of the DEP in the literature. Hansen and Sargent (2007) augue that the DEP values between 0.1 and 0.3 are plausible. In the recent studies, Djeutem and Kasa (2013) show that to match the observed volatility of six U.S. dollar exchange rates (the Australian dollar, the Canadian dollar, the Danish dollar, the Japanese yen, the Swiss franc, and the British pound), the detection error probability should be set between 7.5% to 13.1%. In contrast, Kasa and Lei (2017) and Kasa and Cho (2018) use values above 40% in their models. As a higher DEP means a lower degree of model uncertainty, it suggests that our model does not require unreasonable fears of model misspecification to fit the data.

To further help understand how these parameters influence the four key moments, Figures 4-7 illustrate how these four moments vary with each of the parameters. Among them, Figure 4 is the most important one that shows that the degree of ambiguity aversion has a non-monotonic impact on the relative volatility of production and inventory investment (the top two charts), and also it can alter the sign of the correlations for inventory investment and inventory-to-sales ratio with sales. We have briefly discussed these using the decision rules in the previous section, but now we will dig in deeper.

First, as discussed in the previous section, the U-shaped relationship between μ_{ix} and α (the top right panel) is mainly driven by the effect of α on λ_x. As α increases, the absolute value of λ_x first declines before rising (see Figure 3), which makes inventory investment less sensitive to sales fluctuations in the middle range of α. As this effect dominates other effects (such as those coming from changes in λ_i and λ_Γ), μ_{ix} first declines and then increases with α.

The U-shaped relationship between the volatility of inventory investment and the degree of

[13] Even though the model is exactly identified, the weighting matrix can still play a role because the targeted moment conditions cannot be set exactly equal to zero.

ambiguity aversion also helps explain a similar U-shape between μ_{yx} and α (the top left panel). To better see why, we can use the identity equation (4) to decompose μ_{yx} into:

$$\mu_{yx} = \frac{\operatorname{var}(\Delta i_t)}{\operatorname{var}(x_t)} + 2\frac{\operatorname{cov}(\Delta i_t, x_t)}{\operatorname{var}(x_t)} + 1.$$

Figure 8 clearly shows that the initial decline in variance ratio $\operatorname{var}(y_t)/\operatorname{var}(x_t)$ (when α is small) is mainly driven by the component $\operatorname{var}(\Delta i_t)/\operatorname{var}(x_t)$, while the latter increase (when α is relatively large) is a joint effect of both $\operatorname{cov}(\Delta i_t, x_t)$ and $\operatorname{var}(\Delta i_t)/\operatorname{var}(x_t)$ which both increase with α on the right side of the figure.

Turning to the bottom left panel, two factors help explain why an increase in α could switch the sign of the correlation between inventory investment and sales. Figure 9 plots how different parts of this correlation changes with α based on the definition $\rho_{ix} = \operatorname{var}(\Delta i, x)/(\sigma_{\Delta i}\sigma_x)$. (Notice that σ_x is given so we only plot the other two components.) This clearly shows that the change in the sign of the correlation is due to the same change in the covariance term. As we explained in the previous section, this sign switch is due to the change in λ_x which increases from a negative value to a positive value as α rises. Furthermore, the jump in the magnitude basically reflects the fact that both $|\operatorname{cov}(\Delta i_t, x_t)|$ and $\sigma_{\Delta i}$ drop to zero while the latter declines faster than the former, making the ratio to jump up when the sign switches.

On the correlation between the inventory-to-sales ratio and sales, the signs of inventories generated by the model matters. In the rational expectations (RE) case, when there is a positive cyclical shock to sales, the model-generated inventory stock is utilized as a cushion to the sales shock and decreases, consistent with the counter-cyclical inventory investment result. As we will show in Section 5.3, without the stock-out avoidance assumption the RE model generates negative inventories in most periods. As the reduction in inventories is less proportional than the increase of the sales level, we find that the inventory-sales ratio actually increases after the shock, suggesting a procyclical inventory-sales ratio. As shown in the left panel of Figure 10, after a positive sales shock in the middle of periods, the level of sales increases, while that of inventories decreases, and the overall inventory-sales ratio increases. The right panel of Figure 10 clearly shows that a positive cyclical sales shock increases the firm's inventories, though the increase is proportionally less than the increase of the sales level, and the inventory-sales ratio displays counter-cyclicality. Here the bottom line is, the presence of Knightian uncertainty reverses the cyclicality of inventory investment without making the firm accumulate too much inventory, and preserves the right signs of inventories at the same time. These results guarantee a right sign on the correlation between the inventory-sales ratio and sales.

Figures 5-7 demonstrate how the key moments are influenced by the other three parameters. Figure 5 shows that when the cost shock is more persistent, firms will produce more in the current period compared to a more transitory shock, which leads to more volatile output, a higher inventory-sales variance ratio, and more procyclical inventories. In addition, the inventory-sales ratio is less countercyclical as firms have a stronger incentive to accumulate inventories in the face of persistent cost shocks.

Figure 6 illustrates that if inventory management is more costly, firms reduce inventory holdings and use production to meet unexpected demand shocks. As a result, $\operatorname{var}(y)/\operatorname{var}(x)$ decreases towards 1, and $\operatorname{var}(\Delta i)/\operatorname{var}(x)$ decreases to 0, so in the limit firms hold no inventories and production adjusts one-to-one to meet the sales demand. We can also see from Figure 6 that inventory investment is always procyclical as α_i/α_y varies, and the correlation between inventory investment and sales rises, driven by the lower volatility of inventory investment. In addition, the inventory-sales ratio becomes less countercyclical as inventory holding plays a less and less important role.

Finally, we can see from Figure 7 that when the volatility of the cost shock innovation increases, it has similar effects as that of α and ρ_c on the four moments. Overall, these figures confirm that the selected four moments are informative about the four parameters we want to estimate.

4.2 Evaluation of Models' Key Performance

We now evaluate our RB model's quantitative ability to fit the data, and compare it explicitly to the standard RE model. We report the results in Table 4. The first column reports the empirical evidence during the $1967-2016$ period, the second column reports the RE model's predictions, and the third column reports the RB model's results.

The first row compares the different models' predictions for the relative volatility of production to sales. It shows that the RB model generates a relative volatility of production to sales significantly above 1 as in the data, which solves the production-smoothing puzzle. In contrast, the RE model predicts a value close to 1. As reported in the previous section, the required degree of model uncertainty, as measured by the DEP, is 0.22, which means that firms in our RB model economy face a probability of 0.22 of being unable to distinguish the true sales and cost shock processes. We have already noted that this level of model uncertainty seems reasonable given the literature.

The second row compares the models' predictions for the relative volatility of inventory investment to sales. The RE model significantly overpredicts this ratio, while the RB model generates a ratio three times smaller and is very close to the empirical counterpart in the data. As we explained in the previous section, the RB model is able to reduce the volatility of inventory investment because it reduces the size of the response of inventory investment to sales (λ_x) when the degree of ambiguity aversion starts to increase from 0 (before it reaches the turning point).

The third row compares the models' predictions for the correlation between inventory investment and sales. As explained before, both the PS and PCS models fail to match this fact; the RE model predicts the wrong sign of the correlation between inventory investment and sales. In contrast, the RB model generates the correct sign and a size very close to the empirical counterpart. Again, the change from a negative sign to a positive sign relies on the fact that an increase in ambiguity aversion can potentially make production to change more than sales (following a demand shock) and thus causes inventory investment to move in the same direction as sales.

Fourth, our RB model produces a countercyclical inventory-to-sales ratio, while RE models cannot. The explanation is similar to the previous discussion.

5 RE vs RB

In this section, we will further explain why the RB model outperforms the RE model. First, we will show that the RE model cannot generate a positive correlation between inventory investment and sales. In a validation test, we illustrate how incorporating model uncertainty can improve the results. In addition, we will use a simplified model to explain why allowing the cost shock to be arbitrarily large has the potential to push the relative volatility of production to sales to a level greater than 1. In the last part of this section, we compare our theory with the stockout avoidance motive.

5.1 RE Model's Prediction on the Inventory-Sale Correlation

We can analytically illustrate that the RE model has the following properties.

Proposition 3 *The RE model always implies a negative correlation between inventory investment and sales. In the RE model, the inventory policy is:*

$$i_t = \lambda_i i_{t-1} + \lambda_x x_t - \lambda_\Gamma \Gamma_t, \tag{15}$$

where $\lambda_i = \lambda_1$, $\lambda_x = -\frac{\lambda_1(R-\rho)}{R-\rho\lambda_1}$, *and* $\lambda_\Gamma = \frac{\lambda_1(R-\rho_\Gamma)}{R-\rho_\Gamma\lambda_1}$. *The production policy is:*

$$y_t = \mu_i i_{t-1} + \mu_x x_t - \mu_\Gamma \Gamma_t, \tag{16}$$

where $\mu_i = \lambda_1 - 1 < 0, \mu_x = 1 + \lambda_x > 0, \mu_\Gamma = \lambda_\Gamma$, x_t *is governed by (6),* Γ_t *is governed by (2), and* λ_1 *is the smaller root* $(0 < \lambda_1 < 1)$ *satisfying* $\lambda_1 + R/\lambda_1 = 1 + R + R\alpha_i/\alpha_y$.

Proof. We provide detailed derivation of the policy rules (15) and (16) in Online Appendix A. As $\lambda_1 < 1$ and $\rho_\Gamma \leq 1$, we have $R - \rho_\Gamma\lambda_1 > R - \rho_\Gamma > 0$. We then have $\lambda_x = -\frac{\lambda_1(R-\rho)}{R-\rho\lambda_1} < 0$. Thus, policy rule (15) suggests that an increase in sales leads to a decline in inventory investment. ■

Now note that the inventory policy, (15), can be written as a combination of two AR(2) processes:

$$i_t = \lambda_x \frac{\varepsilon_t}{(1-\rho \cdot L)(1-\lambda_i \cdot L)} + \lambda_\Gamma \frac{w_t}{(1-\rho_\Gamma \cdot L)(1-\lambda_i \cdot L)}, \tag{17}$$

which shows that the stochastic properties of inventories are governed by the following two channels: i) ρ and ρ_Γ, which govern the external propagation mechanisms, and ii) $\lambda_i = \lambda_1$, which is an internal and endogenous propagation mechanism determined by the relative importance of the production smoothing motive to the backlog cleanup motive, α_i/α_y. Also note that λ_1 is determined by R and α_i/α_y, but is independent of the value of ρ and ρ_Γ. In addition, as can be seen in (15) and (17), adding a cost shock does not change the response of the current inventory stock to sales and lagged inventory stocks, so it does not change the covariance of inventory investment and sales. Accordingly, compared with the production-level smoothing model, adopting the production-cost smoothing specification affects the value of $\operatorname{var}(\Delta i_t)$, but has no effect on the inventories-sales correlation; as a result, moderate cost shocks cannot generate the observed higher variance ratio of production to sales. Finally, since $\lambda_x < 0$ in this case ($R - \rho_\Gamma > 0$), we can see the the contemporaneous covariance of inventory and sales will always be negative, independent of the cost shock process. The production policy, (16), implies that production is negatively correlated with the lagged level of inventories and is positively correlated to the current level of sales. If firms have plenty of inventory stock from the last period, they do not need to increase current period production. Similarly, if demand is high in the current period, firms will produce more.

Table 5 shows a validation test based on the RE model. Specifically, we estimate the three unknown parameters, $\Theta = \{\rho_c,\ \Psi,\ \alpha_{iy}\}$, by using the two step GMM procedure, so that the RE model-generated moments matches the three key moments: (1) the relative volatility of production to sales ($\operatorname{var}(y)/\operatorname{var}(x)$), (2) the contemporaneous correlation between inventory investment and sales ($\operatorname{corr}(\Delta i, x)$), and (3) the correlation between inventory-sales ratio and sales ($\operatorname{corr}(i/x, x)$) as near as possible. The second column gives estimated parameters, and the last column shows the RE model-generated results. It can be seen that the RE model can merely generate an output-sales variance ratio around 1.03, while inventory investment is countercyclical, and the inventory-sales ratio is procyclical. All these results are far apart with the empirical evidence. Note that we need almost four times larger volatility of the innovation to the cost shock calibrated in the RB model to generate the output-sales variance ratio of 1.03. Then we compare if the RB model can outperform the original RE model. Figure 12 shows how the RE model's performance improves when the degree of RB rises (i.e., α rises). In the figure, the $\alpha = 0$ case corresponds to the RE model. We can see that as the degree of RB rises, the firm's optimal production and inventory decisions generate a higher output-sales variance ratio, and the comovement of inventory investment and the inventory-sales ratio switches signs and gets closer to the empirical counterpart.

5.2 The Cost Shock and Relative Volatility of Production to Sales

To better understand the inventory and output dynamics, we consider the special case that $\rho = 0$, that is, the sales process is iid. First, given (15), we have:

$$\Delta i_t = \lambda_i \Delta i_{t-1} + \lambda_x \Delta x_t + \lambda_\Gamma \Delta \Gamma_t,$$

Taking the unconditional variance on both sides of this equation yields:

$$\text{var}(\Delta i_t) = \frac{2\lambda_x^2}{1+\lambda i} \text{var}(x_t) + \frac{2\lambda_\Gamma^2}{1+\lambda_i} \text{var}(\Gamma_t). \tag{18}$$

The covariance between the change in inventories and sales is the same as that without the cost shock and can be written as:

$$\text{cov}(\Delta i, x_t) = \text{cov}\left(\frac{\lambda_x \Delta x_t}{1-\lambda_i L}, x_t\right) = \lambda_x \text{var}(x_t), \tag{19}$$

The variance of output can thus be written as:

$$\begin{aligned} \text{var}(y_t) &= \text{var}(x_t) + \text{var}(\Delta i_t) + 2\,\text{cov}(\Delta i_t, x_t) \\ &= \left(1 + \frac{2\lambda_x^2}{1+\lambda_i} + \frac{2\lambda_\Gamma^2}{1+\lambda_i}\frac{\text{var}(\Gamma_t)}{\text{var}(x_t)} + 2\lambda_x\right)\text{var}(x_t), \end{aligned} \tag{20}$$

which means that the condition for the output-sales variance ratio to be larger than one is:

$$\frac{2\lambda_x^2}{1+\lambda_i} + \frac{2\lambda_\Gamma^2}{1+\lambda_i}\frac{\text{var}(\Gamma_t)}{\text{var}(x_t)} + 2\lambda_x > 0, \tag{21}$$

which reduces to

$$-\frac{\lambda_1}{1+\lambda_1} + \frac{\lambda_\Gamma^2}{1+\lambda_1}\frac{\text{var}(\Gamma_t)}{\text{var}(x_t)} > 0 \tag{22}$$

when $\rho = 0$.

It is clear from (22) that if the variance of cost shock is large enough relative to that of the sales process, we can obtain $\text{var}(y_t) > \text{var}(x_t)$. Intuitively, as pointed out in Eichenbaum (1989), incorporating cost shocks increases the amount of uncertainty in the supply side, while leaving the demand side unchanged. In this case, the firm will use inventories to exploit the intertemporal substitution possibilities in production, making production more volatile. For example, suppose that firms observe a positive persistent cost shock. If the cost shock is a mean-reverting AR(1) process, firms will anticipate high marginal costs in subsequent periods, though not as high as in the current period. Then, at present, firms have an incentive to produce less and wait for costs to fall; the more persistent the cost shock, the less incentive the firm has to wait and see. If cost shocks are iid, this mechanism is very strong and output becomes highly volatile. However, an unappealing feature of this solution is that assuming a highly volatile cost shock makes the result assumption-driven – output is volatile effectively because it is volatile, and the model itself plays little to no role.

Another unappealing feature of the cost shock theory is that, as can be observed from Equation (19), the coefficient of the inventory decision rule on x_t, λ_x, is always negative in the standard RE production-smoothing model. The following argument is based on the results in Section 5.1. In a linear-quadratic model with independent demand and cost shocks, the response of firms' production to a demand shock is not affected by the cost-smoothing motive, so adding cost shocks does not change the coefficient on the demand shock in the production decision rule. In addition, according to the identity equation, as inventory investment is the residual term of output and sales, the response coefficient of inventory to sales also remains unchanged. As a result, adding a cost shock itself cannot generate a right sign for the relation between inventory investment and sales.

5.3 Comparison With the Stockout Avoidance Motive

While our model shows that the uncertainty due to concerns about model misspecification can significantly improve the PCS model's predictions on these important aspects of the data, Wen (2005) shows that incorporating the stockout avoidance motive also helps explain the data along several dimensions.[14] The way Wen (2005) models the stockout avoidance motivation is to assume that firms cannot allow inventory to fall below zero (i.e., $i_t \geq 0$), which means firms want to avoid the situation where they must produce more to meet current demand. Our model with model uncertainty can endogenously generate this outcome: as firms have stronger concerns about model misspecification, their production decisions become more responsive to demand shocks and therefore accumulate more inventories; consequently, it is more likely to avoid the negative inventory levels (although the model does not explicitly impose any restrictions on negative inventories).[15]

We first provide details about how to simulate the fraction of inventory stocks given different degree of model uncertainty. Given the values of robust parameter α and other set of parameters, we can simulate the model 1000 times and caluculate the average fraction of negative inventories in all periods. Like in previous section, in each simulation path we again simulate according to (2)

[14] The discussion on the stockout avoidance motive goes back to Kahn (1987) and Maccini and Zabel (1996).

[15] Note that allowing for negative inventory stocks need not require that stocks are actually negative (which is of course impossible); rather, one can interpret a negative inventory stock as indicating future orders in excess of current stocks.

and (6), and use the numerical decision rules, (11) and (12) to obtain the level of inventory stocks. Particularly, we set the inital value of inventory stocks here at the mean value of history sales here, because in the data total business and manufacturing inventory/sales ratio is around 1. Also, we do not add back the trend of log sales when calculating the level of sales, because otherwise when taking exponential it will dominate the zero mean cost shocks and make the signs of sales positive. Figure 13 plots the average share of negative inventory based on simulated series for different values of the RB parameter, α.[16]

As Figure 13 shows, the share of negative inventory is very high if $\alpha = 0$ (the RE model). As α increases, the fraction of negative inventory holdings starts to decrease, though initially it declines very gradually. As α approaches the threshold that leads the response of inventory to sales (λ_x) to change from negative to positive (see the top right panel in Figure 3), the share of negative inventory declines rapidly to a level close to zero. If firms have stronger concerns about model misspecification, they adjust their production more aggressively in response to sales changes and therefore endogenously avoid negative inventory stocks. This result has the flavor of precautionary savings; with higher effective risk aversion, firms naturally hold buffer stocks of inventory and avoid stockouts, even though having a negative inventory is not explicitly punished.

In the literature such as Kahn (1987), Eichenbaum (1989), Ramey (1991), and Ramey and West (1999), there is another way to explicitly model the stockout avoidance motive in the PS and PCS models. Specifically, they assume that the inventory holding cost takes the following form:

$$H(i_t) = \frac{1}{2}\alpha_i \left(i_{t-1} - \alpha_x x_t\right)^2, \tag{23}$$

where i_{t-1} is the inventory level at $t-1$, x_t is the amount of sales in t, and $\alpha_x > 0$ governs the stockout avoidance motive. This motive induces the firm to hold inventories even without production cost considerations, and (23) embodies both inventory holding and backlog costs. If $\alpha_x = 0$, it can be interpreted as an inventory holding cost function. In contrast, if $\alpha_x \neq 0$, this function reflects stockout (backlog) as well as inventory holding costs, and thus captures a revenue-related motive for holding inventories (as α_x increases, stockout costs rise relative to backlog costs). Stockout costs arise when sales exceed the available stock. In other words, the higher the stock of inventories, the less likely is a stockout and the lower are stockout costs.

Figure 14 plots the average share of negative inventory based on simulated series for different values of the degree of the SOA motive, α_x. We can see from this figure that when α_x is sufficiently large ($\alpha_x > 1.64$), the average share of negative inventory in the model with the SOA motive drops to zero. It is worth noting that as summarized in Ramey and West (1999), the median estimate

[16] The average is based on 1000 simulations. We have checked that increasing the number of simulations does not change the results.

of α_x varies from 0.4 (Ramey 1991) to 1.15 (Eichenbaum 1999), which are well below the required critical value, 1.64, in the model.

6 Conclusions and Future Research

In this paper, we construct an ambiguity aversion of the PCS model to study the joint dynamics of production, inventories, and sales. Our calibrated version of the model can explain four facts that previous studies find difficult to account for simultaneously. We also show that the stock-out avoidance motive (Kahn 1987) emerges endogenously in our model. Our analysis suggests that allowing for a moderate degree of ambiguity aversion enables a standard linear-quadratic framework based on Holt et al. (1960) to account for production and inventory dynamics reasonably well.

In our model, rising marginal costs lead firms to use inventories to smooth production. However, this pattern of costs is not necessarily the only assumption in all applications. For example, when wholesalers and retailers purchase goods from manufacturers, products are taken off one shelf, transformed in some way, and put on another shelf. In this situation, it may be more natural to assume that after paying a fixed cost to handle an order, the marginal costs are more likely to be constant or decreasing. These kinds of inventories probably need a different model. As argued in Blinder and Maccini (1991), this technological assumption leads to the so-called (S, s) model of inventory behavior. The economic implications of the (S, s) model differ from those of the PS or PCS models, at least at the individual-firm level. For example, a firm following an (S, s) rule does not have an optimal level of inventory. Instead, it has an optimal range, (S, s), where the upper and lower barriers, S and s, are optimally chosen. Whether model misspecification still plays an important role in such models is a topic we leave for future research.

7 Appendix

7.1 The Equivalence between the Risk-Sensitive and Robust Control Specifications

Following Hansen and Sargent (2010), we assume that the representative firm has a multiplier preference as it ranks plan y_i according to:

$$\min_{\{m_i \geq 0\}_{i=1}^{I}} \sum_{i=1}^{n} m_i \pi_i \left[C(y_t) + H(i_t) + \lambda \ln(m_i) \right]$$

subject to: $\sum_{i=1}^{n} \pi_i m_i = 1$, where $C(y_t)$ and $H(i_t)$ are the production cost and inventory-holding cost functions defined in (1) and (3), respectively. In this setting, $\lambda \in (\underline{\lambda}, \infty)$ is a parameter that, by penalizing choices of m with larger entropy, expresses the firm's concern about possible misspecification of the baseline model governed by the probability distribution, π. The robust inventory model can thus be written as:

$$v(s_t) = \min_{y_t} \left\{ \alpha_y \Gamma_t y_t + \frac{1}{2}\alpha_y y_t^2 + \frac{1}{2}\alpha_i i_t^2 + \beta \max_{m_{t+1}} \mathbb{E}_t \left[m_{t+1} v(s_{t+1}) + \lambda m_{t+1} \ln(m_{t+1}) \right] \right\}, \tag{24}$$

subject to (2), (4), and (6), where s_t is the state vector.

The FOC with respect to m_{t+1} gives:

$$m_{t+1}^* = \frac{\exp(-v(s_{t+1})/\lambda)}{\mathbb{E}_t\left[\exp(-v(s_{t+1})/\lambda)\right]}.$$

Substituting the expression for m_{t+1} into (24) yields:

$$\begin{aligned} & v(s_t) \\ &= \min_{y_t} \left\{ \begin{array}{c} \alpha_y \Gamma_t y_t + \frac{1}{2}\alpha_y y_t^2 + \frac{1}{2}\alpha_i i_t^2 \\ +\beta \mathbb{E}_t \left[\frac{\exp(-v(s_{t+1})/\lambda)}{\mathbb{E}_t[\exp(-v(s_{t+1})/\lambda)]} v(s_{t+1}) + \frac{\lambda \exp(-v(s_{t+1})/\lambda)}{\mathbb{E}_t[\exp(-v(s_{t+1})/\lambda)]} \ln\left(\frac{\exp(-v(s_{t+1})/\lambda)}{\mathbb{E}_t[\exp(-v(s_{t+1})/\lambda)]} \right) \right] \end{array} \right\} \\ &= \min_{y_t} \left\{ \alpha_y \Gamma_t y_t + \frac{1}{2}\alpha_y y_t^2 + \frac{1}{2}\alpha_i i_t^2 - \lambda\beta \ln \mathbb{E}_t \left[\exp\left(-\frac{v(s_{t+1})}{\lambda} \right) \right] \right\} \end{aligned}$$

When λ is set to be $2/\alpha$, we get 7.

7.2 The Expressions for the First Three Moments

Using (12), we can easily obtain that

$$\begin{aligned} \Delta i_t &= \lambda_1 \Delta i_{t-1} - \lambda_1 \Delta x_t + \lambda_\Gamma \Delta \Gamma_t \\ &= \lambda_x \frac{\Delta x_t}{1 - \lambda_i \cdot L} + \lambda_\Gamma \frac{\Delta \Gamma_t}{1 - \lambda_i \cdot L} \\ &= \lambda_x \left(\varepsilon_t + \sum_{k=0}^{\infty} M_k \varepsilon_{t-1-k} \right) + \lambda_\Gamma \left(w_t + \sum_{k=0}^{\infty} M_{\Gamma k} w_{t-1-k} \right), \end{aligned} \tag{25}$$

where λ_i, λ_x, λ_Γ are coefficients of the robust inventory rule in (12). (See Online Appendices A for the detailed derivations of the decision rules.) Using (11) and (25), we can obtain the following explicit expressions for the first three targeted moments:

$$\mu_{yx} \equiv \frac{\text{var}(y_t)}{\text{var}(x_t)} = 1+(1-\rho^2) \left\{ \lambda_x^2 \left(1 + \sum_{k=0}^{\infty} M_k^2 \right) + \lambda_\Gamma^2 \left(1 + \sum_{k=0}^{\infty} M_{\Gamma k}^2 \right) \psi + 2\lambda_x \left[1 + \sum_{k=0}^{\infty} (\rho^{k+1} M_k) \right] \right\}, \tag{26}$$

$$\mu_{ix} \equiv \frac{\text{var}(\Delta i_t)}{\text{var}(x_t)} = (1-\rho^2) \left[\lambda_x^2 \left(1 + \sum_{k=0}^{\infty} M_k^2 \right) + \lambda_\Gamma^2 \left(1 + \sum_{k=0}^{\infty} M_{\Gamma k}^2 \right) \psi \right], \tag{27}$$

and

$$\rho_{ix} \equiv \frac{\operatorname{cov}(\Delta i_t, x_t)}{\sqrt{\operatorname{var}(\Delta i_t)}\sqrt{\operatorname{var}(x_t)}} = \frac{\lambda_x \sqrt{1-\rho^2}\left[1+\sum_{k=0}^{\infty}\left(\rho^{k+1}M_k\right)\right]}{\sqrt{\lambda_x^2\left(1+\sum_{k=0}^{\infty}M_k^2\right)+\lambda_\Gamma^2\left(1+\sum_{k=0}^{\infty}M_{\Gamma k}^2\right)\psi}}, \tag{28}$$

where $M_k = \lambda_i^{k+1} + (\rho-1)\sum_{j=0,j\leq k}^{k}(\rho^j \lambda_i^{k-j})$, $M_{\Gamma k} = \lambda_i^{k+1} + (\rho_\Gamma - 1)\sum_{j=0,j\leq k}^{k}(\rho_\Gamma^j \lambda_i^{k-j})$, and $\psi = \operatorname{var}(w_t)/\operatorname{var}(\varepsilon_t)$ is the relative volatility of the innovation to the cost shock to that to the demand shock.

7.3 Details on the GMM Estimation of the Parameters

First, when we solve the multi-dimensional control problem, (6) is estimated using detrended log sales. The robust version of the model is then solved numerically given the sales process, the identity equation and the cost shock process whose parameters needs to be estimated. To be more specific, a robustness check estimation as well as the GMM approach are adopted to estimate the parameters:

- In the first approach, we use the identity matrix to weight the mean squared distance. In each loop of the estimation we simulate 1000 paths according to (6) and (2), and use the numerical decision rules, (11) and (12) to obtain the level of output and inventory stocks. The inital level of inventory stock is set at zero, and we add back the history trend of log sales to simulated log sales to get level of sales in each path. Using the model generated level data the four relevant moments are calculated in the same way as we get the target moments in each path. That is, we take logs of the moment-generated sales, x and output y, do the HP filtering and construct the related moments. For the inventory data, we define $\Delta i_t = (i_t - i_{t-1})/x_t$, same as the treatment to the data. For the inventory/sales ratio, i_t/x_t, we calculate it as level of inventory stock divided by level of sales. Each time we simulate 596 periods likewise. We take the average of the simulated moments to be the model-generated results. The four unknown parameters are optimally chosen so that the model-generated moments are closest to the data-generated moments using the DIRECT optimization algorithm.[17] The optimal estimates are denoted as MSD estimates as in Table 3.

In the second approach we use a two step GMM estimation. The optimal weighting matrix for the GMM estimation is constructed in two steps:

- First, we substitute the aforementioned mean squared distance estimates into the model and simulate the model 1000 times again. In each path the four relevant moments are calculated

[17] The DIRECT optimization algorithm is a modification to the Lipschitzian optimization algorithm and works much faster than the regular MATLAB optimization. See Jones, Perttunen, and Stuckman (1999) for a discussion.

as what we do in the MSD estimation. We then derive the moments' corresponding variances within 1000 paths. The inverses of the variances will be used as the optimal weights for the corresponding moments in the GMM estimation.

- Second, we do the GMM estimation. In each loop of the estimation algorithm, the model is simulated 1000 times, and then for each simulated path, the four moments mentioned above are calculated. The distance of model generated moments and target moments will now be weighted by the optimal weighting matrix we construct in the first step. The GMM estimates delivers parameters that best match the data moments.

7.4 Calibration of the Detection Error Probability

To provide a more meaningful interpretation of the ambiguity aversion and robust parameter (α), we follow Anderson, Hansen, and Sargent (2003) and Chapter 8 of Hansen and Sargent (2007) to calculate the corresponding detection error probability (DEP). Specifically, we calibrate this model parameter based on a statistical theory of model selection. We can then infer what values of the RB parameter (α) imply reasonable fears of model misspecification for empirically-plausible approximating models. In other words, the model detection error probability is a measure of how far the distorted model can deviate from the approximating model without being discarded.

Let model A denote the approximating model and model B be the distorted model. Define p_A as

$$p_A = \text{Prob}\left(\log\left(\frac{L_A}{L_B}\right) < 0 \middle| A \right), \tag{29}$$

where $\log\left(\frac{L_A}{L_B}\right)$ is the log-likelihood ratio. When model A generates the data, p_A measures the probability that a likelihood ratio test selects model B. In this case, we call p_A the probability of the model detection error. Similarly, when model B generates the data, we can define p_B as

$$p_B = \text{Prob}\left(\log\left(\frac{L_A}{L_B}\right) > 0 \middle| B \right). \tag{30}$$

The detection error probability, p, is then defined as the average of p_A and p_B:

$$p(\alpha) = \frac{1}{2}(p_A + p_B), \tag{31}$$

where α is the robustness parameter used to generate model B. Given this definition, we can see that $1 - p$ measures the probability that econometricians can distinguish the approximating model from the distorted model. Now we show how to compute the model detection error probability in our RB model.

Next, to further calibrate the RB parameter used in our benchmark model, we assume that the RE model is the approximating model, in which the state vector, $s_t = (i_{t-1}, x_t, \Gamma_t)'$, evolves according to:

$$\begin{bmatrix} i_t \\ x_{t+1} \\ \Gamma_{t+1} \end{bmatrix} = A_1 \begin{bmatrix} i_{t-1} \\ x_t \\ \Gamma_t \end{bmatrix} + C \begin{bmatrix} \varepsilon_{t+1} \\ w_{t+1} \end{bmatrix}, \tag{32}$$

where we used the policy function for y_t, (16), obtained by solving the RE model. In contrast, assuming that the distorted model generates the data, s_t evolves according to:

$$\begin{bmatrix} i_t \\ x_{t+1} \\ \Gamma_{t+1} \end{bmatrix} = A_2 \begin{bmatrix} i_{t-1} \\ x_t \\ \Gamma_t \end{bmatrix} + C \begin{bmatrix} \varepsilon_{t+1} \\ w_{t+1} \end{bmatrix}, \tag{33}$$

where we used the policy function for y_t, (11) and robust distortion b_t, (23) in the online appendices, obtained by solving the RB model.

In order to compute p_A and p_B, we use the following procedure:

1. Simulate $\{s_t\}_{t=0}^{T}$ using (32) and (33) a large number of times. The number of periods used in the simulation, T, is set to be the actual length of the data.
2. Count the number of times that $\log\left(\frac{L_A}{L_B}\right) < 0 \Big| A$ and $\log\left(\frac{L_A}{L_B}\right) > 0 \Big| B$ are each satisfied.
3. Determine p_A and p_B as the fractions of realizations for which $\log\left(\frac{L_A}{L_B}\right) < 0 \Big| A$ and $\log\left(\frac{L_A}{L_B}\right) > 0 \Big| B$, respectively.

Once we compute the values of p_A and p_B by simulating the two model economies with a specific α, we can obtain the detection error probability, $p = (p_A + p_B)/2$. To calibrate α, we reverse the procedure and find the unique α for each reasonable p. Figure 11 shows how p varies given the estimated parameters, and can provide a guidance for the selection of the plausible values of the RB parameter. Notice that when $\alpha = 0$, the DEP is 0.5, because the approximating model and the distorted model are identical in this case. In other words, the decision maker does not care about model misspecification and takes the approximation model as given. The DEP falls quickly when α rises with the underlying parameter. The calibrated DEP is 0.22, which means that there is a 22% chance that a likelihood ratio test will improperly select between the approximating and the distorted models. As emphasized in Hansen and Sargent (2007), in the robustness model, p is the deep model parameter governing the amount of model uncertainty, and α reflects the effect of RB on the model's behavior. We also do the robust check and vary the robust parameters. When DEP increases from 0.22 to 0.25, we can see the effect of robustness on benchmark model does not vanish and model generated moments remains in plausible neighborhood of data.

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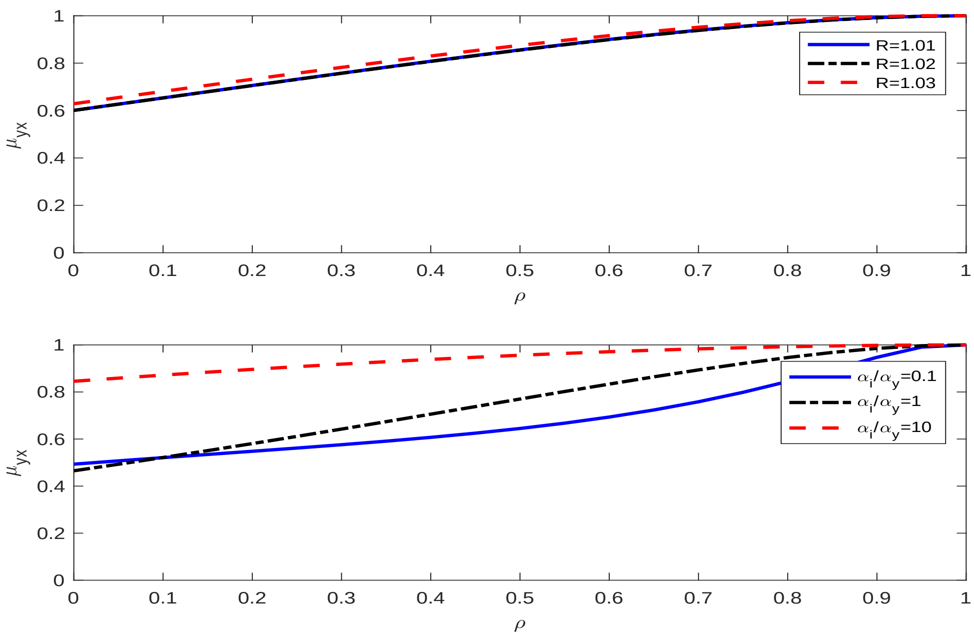

Figure 1: Effects of ρ, R, α_i/α_y on the Relative Volatility μ_{yx}

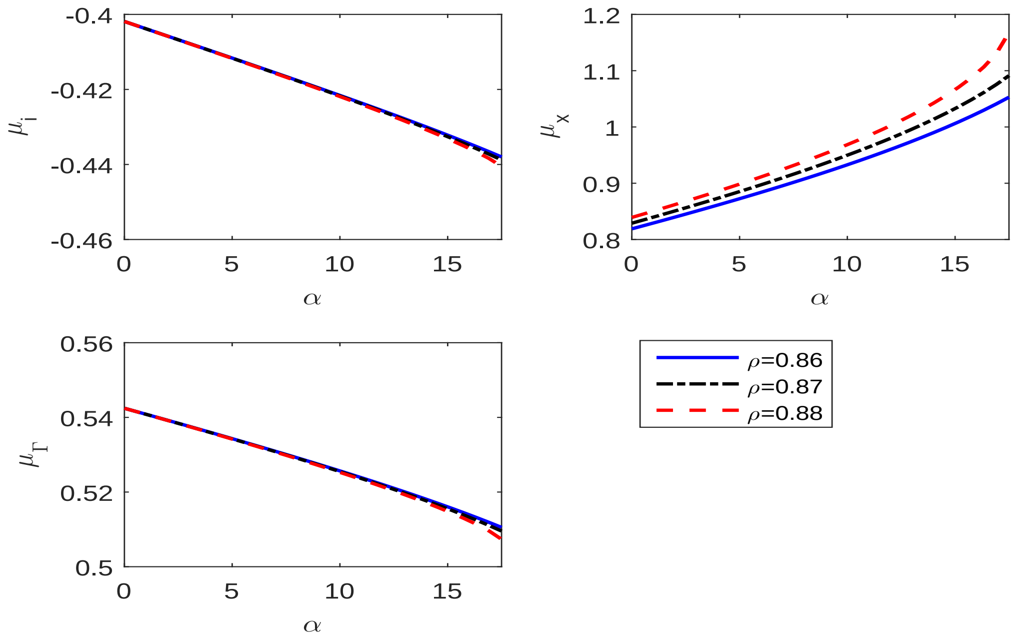

Figure 2: Effects of RB on Production Decisions

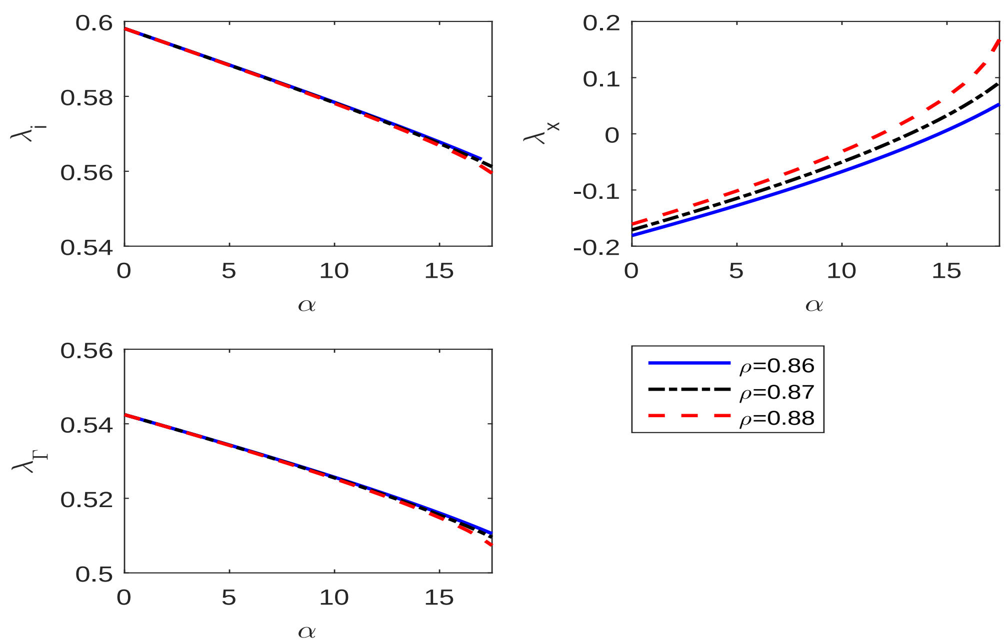

Figure 3: Effects of RB on Inventory Decisions

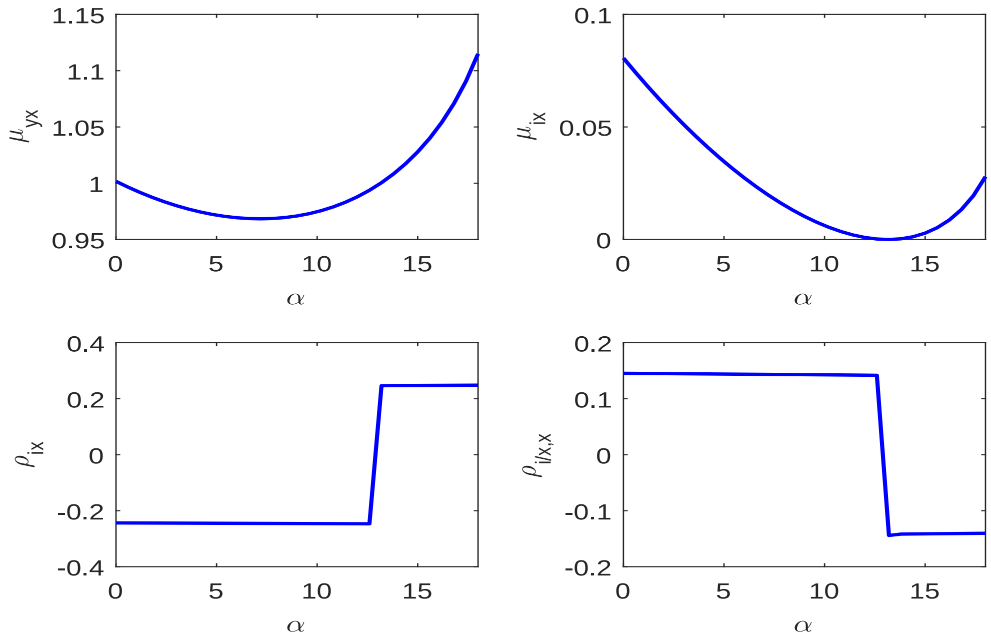

Figure 4: Changes of Moments When Varying the Degree of Model Uncertainty

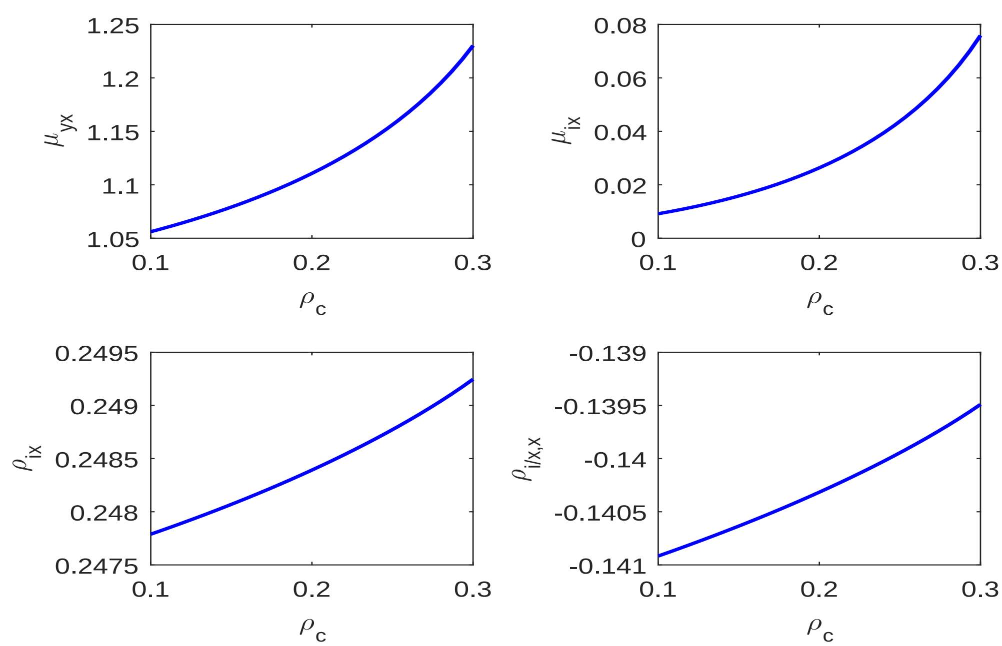

Figure 5: Changes of Moments When Varying the Persistence of Cost Shocks

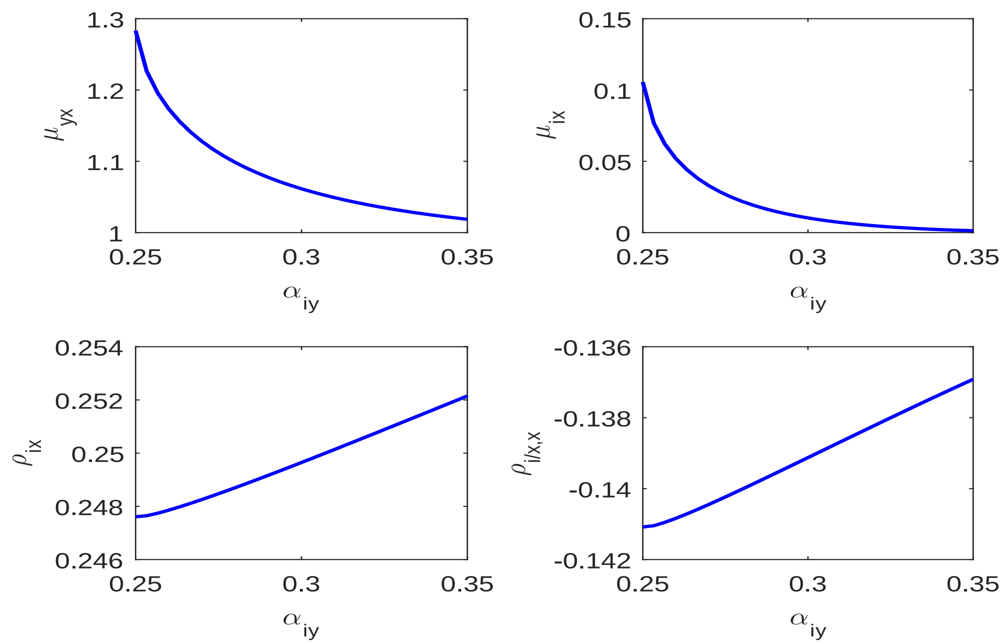

Figure 6: Changes of Moments When Varying the Relative Importance of Inventory Holdings

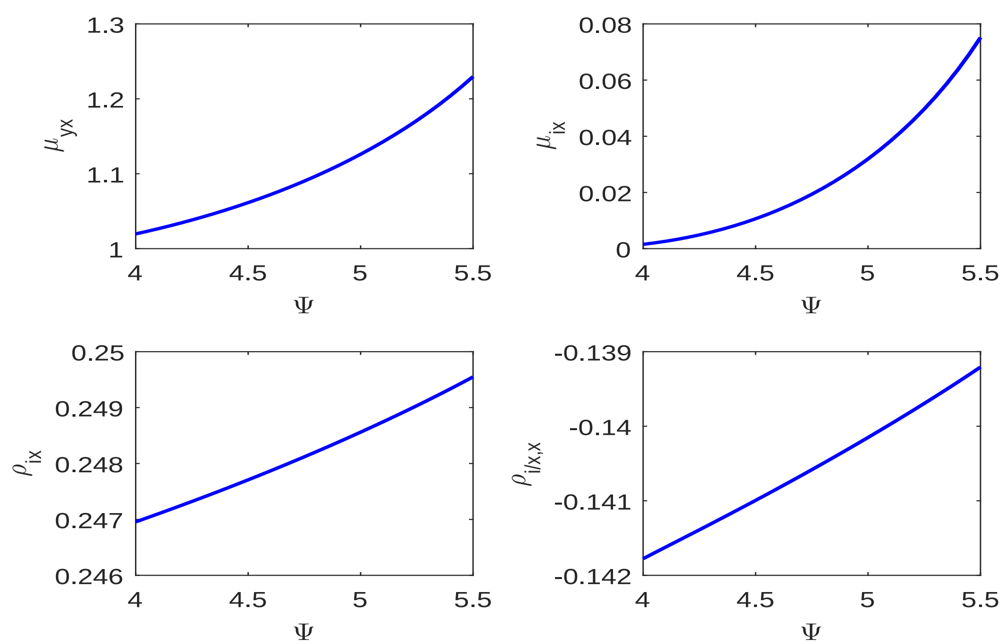

Figure 7: Changes of Moments When Varying the Relative Standard Deviation of Cost Shock Innovation to Sales shocks

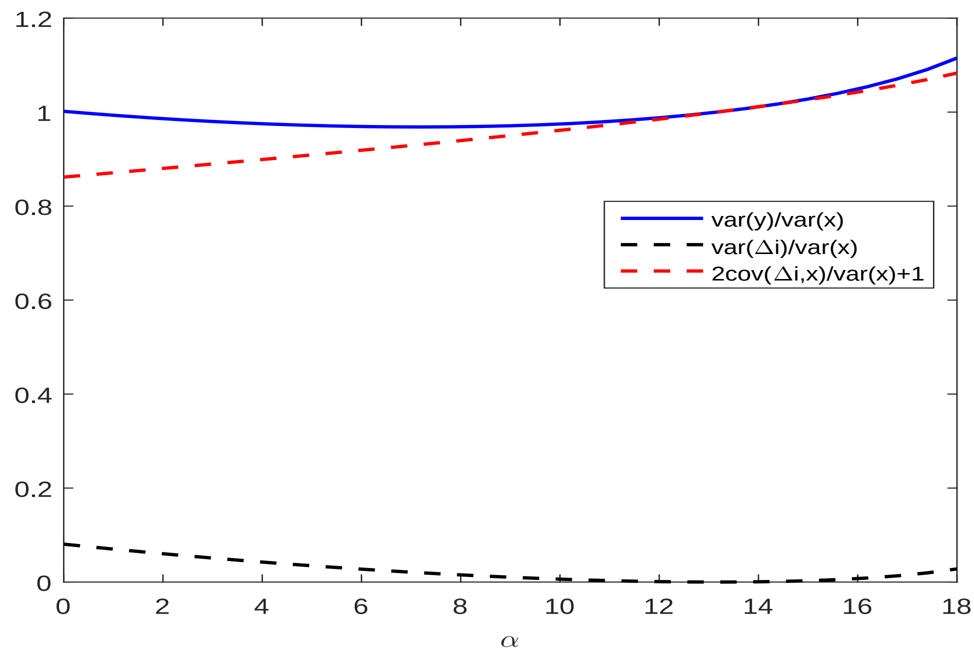

Figure 8: Decomposition of μ_{yx}

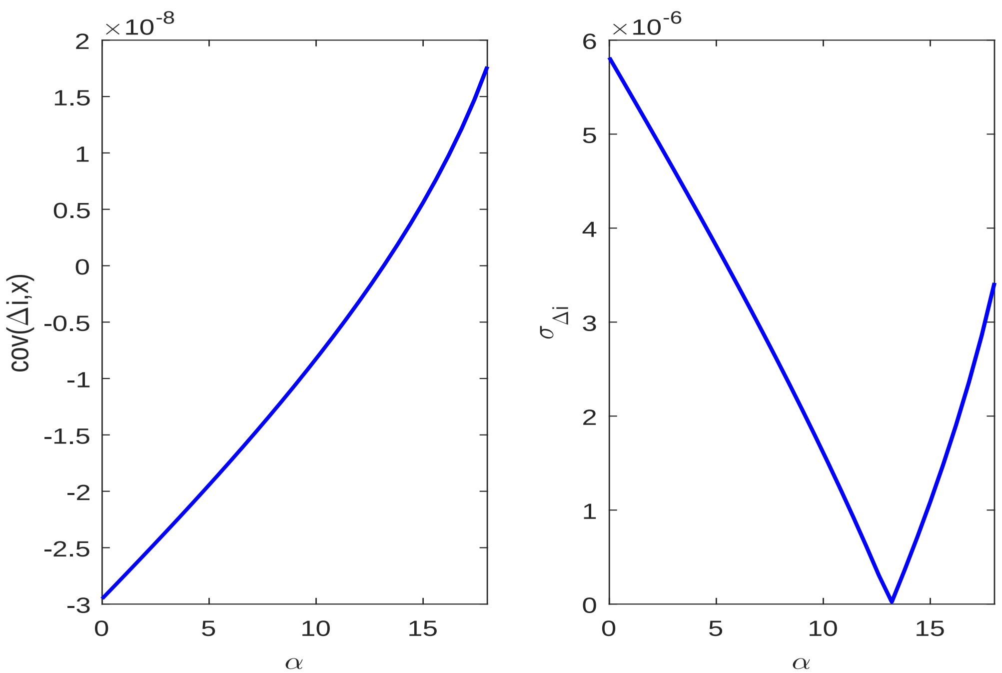

Figure 9: Comparison of $\mathrm{cov}(\Delta i, x)$ and $\sigma_{\Delta i}$

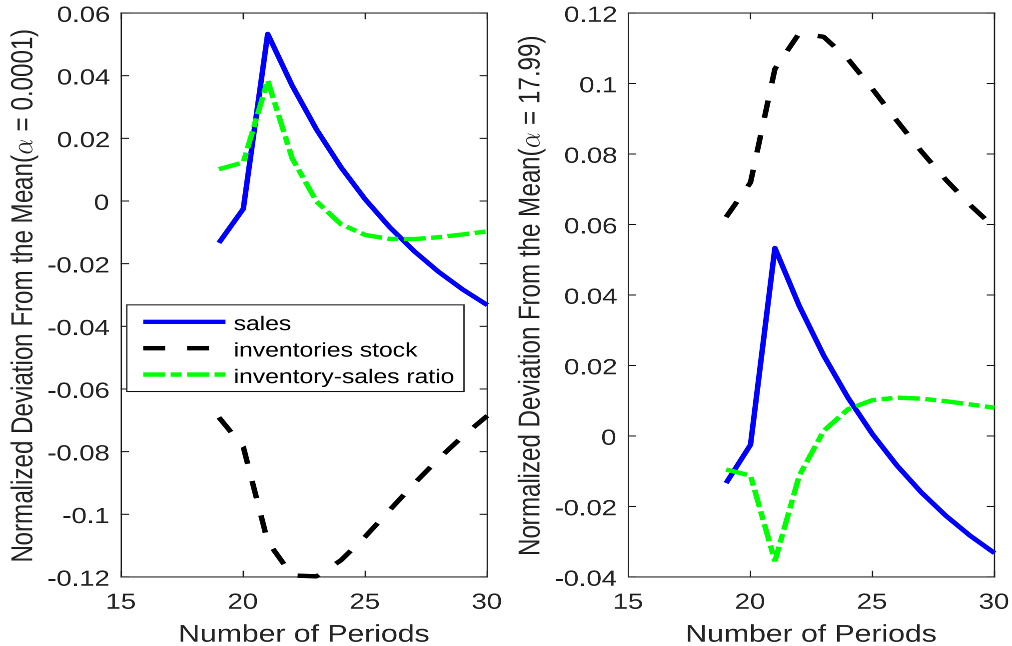

Figure 10: Normalized Deviation from the Mean of Sales, Inventories, and Inventory-sales Ratio After A Sales Shock

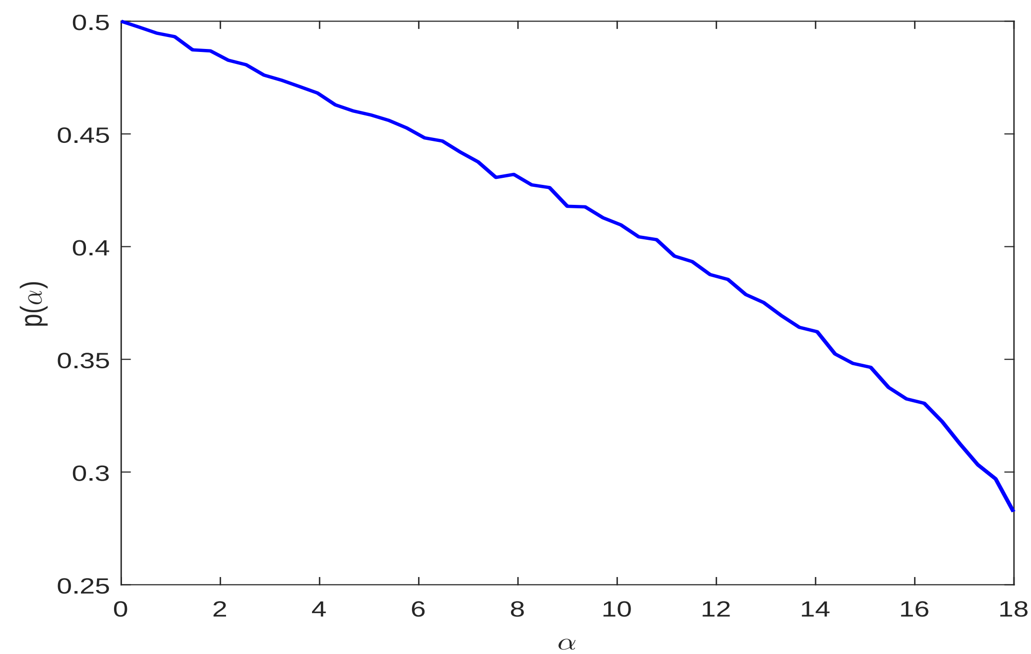

Figure 11: Detection Error Probability (p) as a Function of α in the Manufacturing Sector

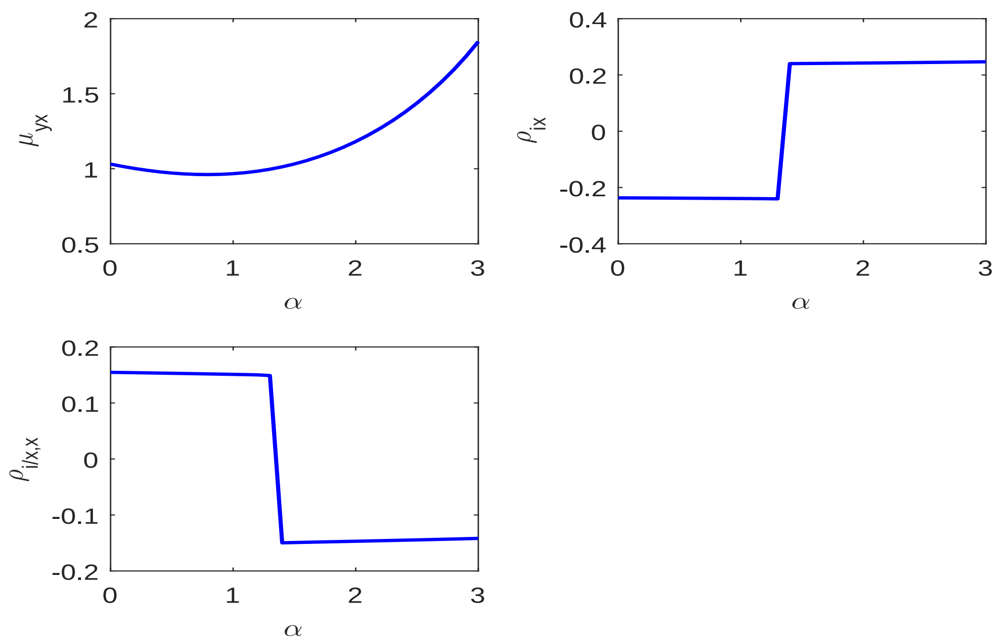

Figure 12: Changes of Moments When Varying the Degree of Model Uncertainty in a Three-Parameter Estimation

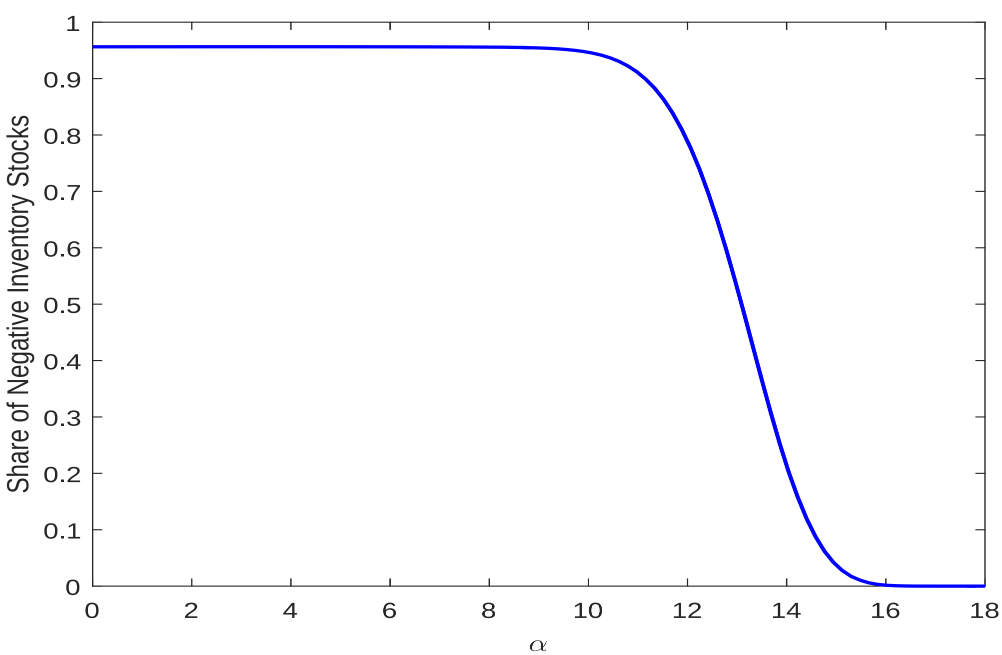

Figure 13: The Simulated Average Fraction of Negative Inventory Stocks When α Rises

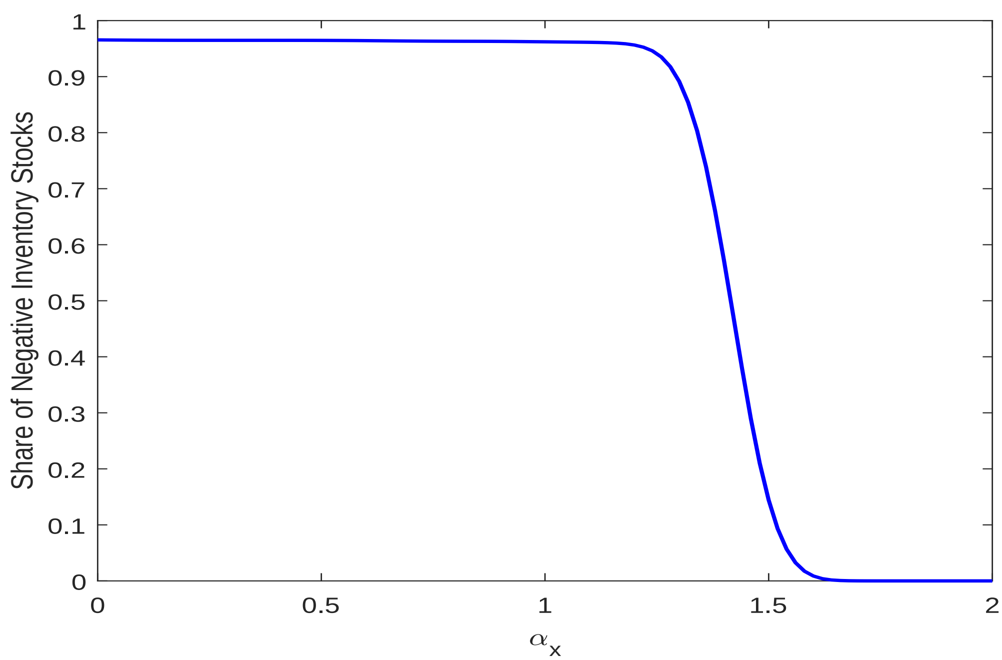

Figure 14: The Simulated Average Fraction of Negative Inventory Stocks When α_x Rises

Table 1: The Joint dynamics of Inventories, Production and Sales

	Whole Sample	Before Financial Crisis
	1967 − 2016	1967 − 2007
Panel A: Manufacturing Sector		
Relative Volatility of Production to Sales (μ_{yx})	1.10 (.0152)	1.10 (.0192)
Relative Volatility of Inventories to Sales (μ_{ix})	0.0283 (.0047)	0.0327 (.0059)
Correlation between Inventory Investment and Sales (ρ_{ix})	0.30 (.0548)	0.29 (.0548)
Correlation between Inventory-Sales Ratio and Sales $\left(\rho_{i/x,x}\right)$	−0.48 (.0641)	−0.49 (.0774)
Panel B: Wholesale Sector		
Relative Volatility of Production to Sales (μ_{yx})	1.26 (.0362)	1.23 (.0339)
Relative Volatility of Inventories to Sales (μ_{ix})	0.133 (.0232)	0.13 (.0265)
Correlation between Inventory Investment and Sales (ρ_{ix})	0.23 (.0648)	0.17 (.0611)
Correlation between Inventory-Sales Ratio and Sales $\left(\rho_{i/s,s}\right)$	−0.38 (.0954)	−0.37 (.1065)
Panel C: Retail Sector		
Relative Volatility of Production to Sales (μ_{yx})	1.24 (.0512)	1.22 (.0556)
Relative Volatility of Inventories to Sales (μ_{ix})	0.248 (.0360)	0.234 (.0383)
Correlation between Inventory Investment and Sales (ρ_{ix})	0.09 (.0766)	0.05 (.0789)
Correlation between Inventory-Sales Ratio and Sales $\left(\rho_{i/s,s}\right)$	−0.12 (.0910)	−0.13 (.0887)

Table 2: Persistence of Sales in the Manufacturing, Wholesale and Retail Sectors

	Whole Sample	Before Financial Crisis
	1967 − 2016	1967 − 2007
Manufacturing Sector	0.87(.0211)	0.84(.0262)
Wholesale Sector	0.85(.0285)	0.84(.0333)
Retail Sector	0.76(.0454)	0.73(.0540)

Table 3: Mean Squared Distance estimates &GMM estimates

Variable	Definition	MSD estimates	GMM estimates
ρ_c	Persistence of cost shock	0.0553	0.2055
Ψ	Volatility of cost shock innovations	0.0041	0.0041
α_{iy}	Relative importance of inventory holding cost	0.2466	0.2740
α	Robust parameter	17.9960	17.9960

Table 4: Model Comparison

	Data	RE	RB ($p = 0.22$)
	1967 − 2016		
μ_{yx}	1.10	1.00	1.11
μ_{ix}	0.028	0.081	0.028
ρ_{ix}	0.30	−0.24	0.25
$\rho_{i/x,x}$	−0.48	0.15	−0.14

Table 5: Performance of the Calibrated RE Model

Parameter	Estimated Value	Moment	Data	Model
ρ_c	0.5542	μ_{yx}	1.10	1.03
Ψ	0.0168	ρ_{ix}	0.30	−0.24
α_{iy}	0.1740	$\rho_{i/x,x}$	−0.48	0.15