

**MARKET MAKERS' SUPPLY AND PRICING OF
FINANCIAL MARKET LIQUIDITY**

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Abstract

This study models the bid-ask spread in financial markets as a function of asset price variability and order flow. The market-maker is characterized as passively accepting orders to buy and to sell a security at the market's prevailing price (plus or minus half the bid-ask spread). The bid-ask spread adjusts to cover market-makers' average costs. The bid-ask spread then varies positively with: the security's price volatility, the volatility of order flow, and the absolute value of the market-maker's net inventory position. Each of these variables increases average cost and hence is priced in the bid-ask spread. Thus market liquidity (varying inversely with the bid-ask spread) declines with increasing price and volume volatility and with increasing size of market-maker net inventory positions. The model hence provides a particularly simple explanation for declining market liquidity during periods of large price movements and trading imbalances that increase the size of market-makers' net inventory.

JEL Classification: G12.

1. Introduction: Market Makers and Market Liquidity

Liquidity in a financial market --- the ability to absorb smoothly the flow of buying and selling orders --- comes from two principal sources: from the depth of buyers and sellers in the market and from the actions of market makers (specialists on the organized exchanges). The market makers' business is acting as intermediaries: buying from public sellers and selling to public buyers. The market maker provides market liquidity --- immediacy services --- and makes his profit from the difference between buying and selling (bid and ask) prices. One simple measure of the liquidity of a financial market at any moment is the bid/ask spread. Liquid markets will be characterized by a narrow spread, illiquid markets by a wide spread. This description applies both cross sectionally and over time. Reduced liquidity is signaled by a broadening of the bid/ask spread.

In actual markets, the behavior of the bid/ask spread varies with market conditions. Increasingly turbulent markets, such as those around October 19, 1987, late August through October 1998, and April 10-14, 2000, are accompanied by significant expansion of the bid/ask spread. In this paper we develop a simple model of market maker pricing that gives an elementary explanation of the variation in market liquidity with variation in market conditions. It derives the market maker's bid/ask spread as the result of the market maker's optimizing behavior. The bid/ask spread is the price of the market-maker's services. The market maker is characterized as a monopolistic competitor, a profit maximizer subject to an (average) zero-profit condition due to the threat of entry. The market maker borrows capital on a perfect (or slightly imperfect) capital market; he faces constant (or increasing) marginal interest rates as his debt position increases.

The principal results of this model then are that the bid/ask spread is an increasing function of two variables: the size of the market maker's net long or short position and (in the case of increasing marginal cost) market price volatility. Both of these variables increase the market maker's average cost of making a market. Hence they are priced in the bid/ask spread.

A distinctive element of the treatment presented here is the simplicity of the market maker's problem. He acts primarily as a reseller of a good subject to the risk of price variability. We purposefully ignore the issues of differential information (informed versus uninformed traders) not because we regard them as unimportant but rather that they may be

independent of the volume and volatility considerations (particularly for the market as a whole) that the present study emphasizes. We suppose the market maker to be risk neutral; he needs no attitude toward risk to determine a bid/ask spread. We consider two cases, first where the capital market is default risk averse and second where the capital market is default risk neutral. In both cases, the bigger the inventory of the market maker, then the higher will be the average capital cost, and the wider will be the bid/ask spread. In the second case, where the capital market is risk averse, the market maker faces an increasing interest rate to finance inventory as his exposure to market risk increases. Therefore, increasing price risk also increases the market maker's average cost, and that cost is priced in the bid/ask spread.

The model hence provides a particularly simple explanation for declining market liquidity during periods of large price movements and trading imbalances (either for the market as a whole or for a single security). Thus, for example, the *Wall Street Journal* of April 17, 2000 reported "by the end of the day [April 14, 2000], it seemed as if the only buyers were NASDAQ market makers and Big Board floor specialists, who have to buy with their own capital when no other buyers can be found. After the NASDAQ's unprecedented Monday-through-Friday losing streak, market makers are holding three times as much stock as usual...the specialist community is stretched." In the setting depicted in this comment, the market maker has typically accumulated a large net position in the security he specializes in; the market maker buys when the public sells. If the price decline continues, both current volatility and the market maker's net position will have increased significantly from their pre-crash levels. This leads to increased average costs which are then priced in the bid/ask spread.

There has been a large theoretical (and empirical) literature on the determinants of bid-ask spreads. The literature largely agrees that the price risk of the underlying asset and the effect of asymmetric information is important in determining the percentage bid-ask spreads (for example, Copeland and Galai, 1983; Glosten and Milgrom, 1985; and Easley and O'Hara, 1987). The literature, however, differs at the role of inventory (Madhavan 2000). In many studies where inventory was one of the considered variables, it ended up as a non-factor in determining the *percentage* bid/ask spreads (for example, Stoll, 1978; Ho and Stoll, 1981). Garman (1976), Amihud and Mendelson (1980), and O'Hara and Oldfield (1986) are a few exceptions. Both Garman's model and Amihud and Mendelson's model

treat the inventory positions of the market makers as following a “birth-and-death” process with constant asset prices, and market makers set their bid/ask prices to minimize the possibility that their inventory positions will reach zero or infinity. Consequently, the bid and ask prices depend on the inventory positions of the market makers, and the percentage bid/ask spreads increase as the gap between the actual and “preferred” positions grow. The O’Hara and Oldfield model, on the other hand, treats the market maker as an investor who maximizes his utility, which is a function of both the profit of making the market and the value of the inventory.²

This essay differs from these papers as our market makers neither target a desired inventory position nor receive utility from the value of their inventories. The market makers in our model can always liquidate their long positions at the market bid price (or close out their short positions at the market ask price). Inventory enters their choices of bid/ask spreads mainly because the costs of carrying the inventory need to be covered by the revenues from making the market under the equilibrium condition of expected zero profit. It is common in the literature to characterize the market maker as setting a bid and an ask price in level terms. The treatment here simplifies the bid/ask spread decision to choosing a percentage of the prevailing asset price.

2. Modeling the Market Maker

We consider the market maker for a single security whose price at date t is denoted P_t . The evolution of price over time is exogenous. The market maker sets a symmetric proportional spread S_t at date t representing his price markup for the ask price and the markdown for the bid price. At date t , he faces a (long, buying) demand volume V_t^l and a (short, selling) supply volume V_t^s . The market presents the market-maker with demand for securities V_t^l , that the market-maker provides. The market presents the market maker with supplies of the security V_t^s that the market maker purchases. The market maker's position (net holding) of the security may be positive or negative. At the start of date t , we denote the position as N_t . The market maker's position in the security evolves over time. At $t+1$,

² By focusing on both bid and ask prices, the study tries to solve two problems in one shot: one is the determination of security price, and the other is the determination of the price of the market maker’s service – the bid/ask spread. Our model, by using the percentage bid/ask spread, separates the two issues and focuses only on the latter, which allows us to simplify the model greatly and thus gain useful insight on the role of inventory.

his position is his position at t increased by his purchases at t and reduced by his sales. That is,

$$N_{t+1} = N_t - V_t^l + V_t^s.$$

The market maker passively accepts all orders to buy and sell. His only strategic action is to adjust the spread S_t .

The market maker starts period t with a cash position M_t , carried over from the previous period. In conducting business at date t , the market maker incurs costs C_t . We denote the market makers' net asset value position at the start of date t as Π_t . Note that Π then represents a stock variable, the inventory valuation. Π does not represent a profit flow. The value of the market maker's position at the beginning of t then is

$$\Pi_t = P_t N_t + M_t.$$

At $t+1$, the value of the position is

$$\Pi_{t+1} = P_{t+1} N_{t+1} + M_{t+1}.$$

The market maker's cash position evolves then as

$$M_{t+1} = (1 + S_t) P_t V_t^l - (1 - S_t) P_t V_t^s + M_t - C_{t+1}.$$

There are two components in C_{t+1} – the cost of providing market-making services. One is mainly the direct cost of trading, such as order taking, order execution, and record keeping. The other component is the cost associated with having to carry inventories of the security in which the market maker is making the market. For simplicity, the first component is ignored here. That is, ordinary marginal costs of record keeping and execution are taken as constant and do not represent a source of variation in cost. Since we are interested in variation in pricing the bid/ask spread and variation in the market makers' average costs, we ignore this constant cost portion. Conceivably, the cost function can differ depending on whether the market maker carries a long or short position though we do not treat this possibility directly below.

3. Market Makers' Pricing: A Zero-Profit Condition

Without modeling explicitly the market structure of market making activities, we assume a zero-profit condition. In the case of NASDAQ, this represents the outcome of ease of entry into market making activities. On the NYSE, this may be taken to represent

the notion of the cost of maintaining an orderly market --- or a normative ceiling (not necessarily zero) on specialist profits.

The zero-profit condition implies³

$$E_t[\Pi_{t+1}] = E_t[M_{t+1} + P_{t+1}N_{t+1}] = \Pi_t = M_t + P_tN_t$$

On average, the market maker assumes no net position on the securities in which he is making a market. As an equilibrium condition then, expected sales and purchases are on average equal. Assume the expected volumes of the buy and sell orders of the security are the same, and denote it as V_0 . Then,

$$E_t[M_{t+1}] = (1 + S_t)P_tV_0 - (1 - S_t)P_tV_0 + M_t - E_t[C_{t+1}] = 2S_tP_tV_0 + M_t - E_t[C_{t+1}]$$

Now assume that the distribution of price and volume at $t+1$ are not correlated. Further, we assume a martingale condition, that the expected mean of the security price at $t+1$ is equal to the price at t . Then

$$E_{t+1}[P_{t+1}N_{t+1}] = E_{t+1}[P_{t+1}]E_{t+1}[N_{t+1}] = P_t(N_t + V_0 - V_0) = P_tN_t$$

Therefore, the zero profit condition implies

$$E_t[M_{t+1} + P_{t+1}N_{t+1}] = 2S_tP_tV_0 + M_t - E_t[C_{t+1}] + P_tN_t = M_t + P_tN_t,$$

or

$$E_t[C_{t+1}] = 2S_tP_tV_0.$$

In other words, at the market equilibrium, the market maker will set the bid-ask spread so that

$$S_t^* = \frac{E_t[C_{t+1}]}{2P_tV_0}.$$

This expression is the cornerstone of this line of research. The market maker adjusts the bid/ask spread at any moment to cover expected (variable average) costs at expected trading volume. The market maker pursues expected average cost pricing in a variable stochastic environment.

4. The Quadratic Cost Case and the Absolute Value Cost Case

In order to derive a prediction from the pricing model above, we must specify the form of the cost function C_{t+1} . How does the size of the market maker's net position, in the

³ This expression could be presented including a time discount factor, without changing the character of the results.

security in which he makes a market, affect his average costs? Holding the trading inventory N_{t+1} requires financing, and the average cost of capital may vary with the size of the market maker's position.⁴ A risk premium may be added to interest rates on lending to a market maker whose position is increasingly leveraged. Thus we suggest the specification:

Case 1: Quadratic Cost. $C_{t+1} = a(P_{t+1}N_{t+1})^2$.

Then

$$\begin{aligned} E_t[C_{t+1}] &= aE_t[((P_t + \Delta P_{t+1})N_{t+1})^2] = a(P_t^2 + \sigma_p^2)E_t[(N_t + V_{t+1}^l - V_{t+1}^s)^2] \\ &= a(P_t^2 + \sigma_p^2)(N_t^2 + 2\sigma_v^2 + 2V_0^2) \end{aligned}$$

Thus,

$$S_t^* = \frac{a}{2PV_0} (P_t^2 + \sigma_p^2)(N_t^2 + 2\sigma_v^2 + 2V_0^2).$$

Hence, in the quadratic cost case, we find that the market maker's bid/ask spread varies positively with price risk, σ_p^2 , with trading volume risk, σ_v^2 , and the size of the market maker's trading inventory exposure N_t^2 . As markets become more volatile in price or volume, the bid-ask spread expands. As the market maker's exposure --- embodied in the (squared) size of his inventory --- expands, so does the bid-ask spread.

Case 2: Absolute value cost. $C_{t+1} = a | P_{t+1}N_{t+1} |$. Then,

$$E_t[C_{t+1}] = aE_t[P_{t+1}]E_t[|N_{t+1}|] = aP_tE_t[|N_{t+1}|].$$

However, now we need to know the distribution functions of buy and sell volumes to calculate the mean of N_{t+1} since

$$\begin{aligned} E_t[|N_{t+1}|] &= E_t[|N_t - V_{t+1}^l + V_{t+1}^s|] \\ &= \int_{N_t - V_{t+1}^l + V_{t+1}^s \geq 0} (N_t - V_{t+1}^l + V_{t+1}^s) dv^l dv^s - \int_{N_t - V_{t+1}^l + V_{t+1}^s < 0} (N_t - V_{t+1}^l + V_{t+1}^s) dv^l dv^s \end{aligned}$$

under the condition that buy and sell volumes are independent and identically distributed.

Further, assume they both follow uniform distributions. That is,

⁴ The unit time interval is most appropriately conceived as a single trading day. At the close of trade, financing costs are incurred on the market maker's net position. The market maker adjusts the spread during the trading day to attempt to optimize his ability to cover costs on the closing inventory position.

$f(v) = \frac{1}{2\sigma_v}$ in the range of $V_0 - \sigma_v \leq v \leq V_0 + \sigma_v$, and zero otherwise.⁵

Then the relationship between $E_t[|N_{t+1}|]$ and $|N_t|$ is nonlinear when the value of $|N_t|$ is relatively small.⁶ However, for large enough $|N_t|$, (when $|N_t| \geq V_0 + \sigma_v$), it is proved in Appendix 1 that $E_t[|N_{t+1}|] = |N_t|$.⁷

That is,

$$S_t^* = \frac{a}{2V_0} |N_t|.$$

In the absolute value average cost case we find that the bid-ask spread varies directly with the extent of the market-maker's trading inventory exposure to market risk. The bigger the market-maker's position (in absolute value), the bigger the spread.⁸

5. Conclusion

A market maker's expected average costs increase with the size (absolute value) of his inventory position; the cost of financing inventory increases with its size. Further, in the case where the market maker faces increasing marginal financing cost, expected average costs increase with security price volatility. These costs, in equilibrium, must be covered out of the bid/ask spread. Consequently, the bid/ask spread varies directly with security price volatility and with the size of the market maker's trading inventory position. Since an increasingly turbulent asset market is characterized by imbalance of trading demand and supply and by increasing price volatility, this model suggests that (and provides an explanation why) such a market is likely to be accompanied by a deterioration in market liquidity.

⁵ The mean of the distribution is V_0 and the variance is $\sigma_v^2/3$.

⁶ See Appendix 2 for examples.

⁷ Our setting thus leads to the conclusion that when the initial inventory position is large, the time series of inventory position has a unit root, which may or may not be consistent with the empirical evidence (Hasbrouck and Sofianos, 1993). However, this result should not be taken literally, as the focus of this paper is not the statistical property of the inventory and thus, the model is highly stylized.

⁸ Allowing the correlation between volume and price distributions is likely to strengthen the positive relation between the inventory position and bid/ask spread as a sudden surge of inventory should signal negative price changes are more likely, which makes the existing inventory even more costly to the market maker.

Appendix 1: The Proof of Case 2.

In this appendix, we prove that in section 4, case 2, $E_t[|N_{t+1}|] = |N_t|$ for sufficiently large initial inventory position, i.e., $|N_t| \geq V_0 + \sigma_v$.

$$\text{The zero-profit condition implies: } S_t^* = \frac{E_t[C_{t+1}]}{2P_tV_0}.$$

If the cost increases with the absolute value of the inventory, i.e., $C_{t+1} = a|P_{t+1}N_{t+1}|$, then under the assumption that price and transaction volumes are independently distributed, $E_t[C_{t+1}] = aE_t[P_{t+1}]E_t[|N_{t+1}|] = aP_tE_t[|N_{t+1}|]$ as price is always positive.

Now assume that buy and sell volumes are independent and identically distributed, then

$$\begin{aligned} E_t[|N_{t+1}|] &= E_t[|N_t - V_{t+1}^l + V_{t+1}^s|] \\ &= \int_{N_t - v_{t+1}^l + v_{t+1}^s \geq 0} (N_t - v_{t+1}^l + v_{t+1}^s) dv_{t+1}^l dv_{t+1}^s - \int_{N_t - v_{t+1}^l + v_{t+1}^s < 0} (N_t - v_{t+1}^l + v_{t+1}^s) dv_{t+1}^l dv_{t+1}^s. \end{aligned}$$

Further assume they both follow uniform distributions:

$$f(v) = \frac{1}{2\sigma_v} \text{ in the range of } V_0 - \sigma_v \leq v \leq V_0 + \sigma_v, \text{ and zero otherwise. As the volume cannot}$$

be negative, it is necessary that $V_0 \geq \sigma_v$. Then (suppress the subscripts in v_{t+1}^l and v_{t+1}^s)

$$\begin{aligned} E_t[|N_{t+1}|] &= \int_{N_t + v^s \geq 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l \leq N_t + v^s, V_0 - \sigma_v \leq v^l \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\ &+ \int_{N_t + v^s < 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int (0) dv^l \\ &- \int_{N_t + v^s \geq 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l > N_t + v^s, V_0 - \sigma_v \leq v^l \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\ &- \int_{N_t + v^s < 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{V_0 - \sigma_v}^{V_0 + \sigma_v} (N_t - v^l + v^s) dv^l. \end{aligned}$$

That is,

$$\begin{aligned}
E_t[|N_{t+1}|] &= \int_{N_t+v^l \geq 0, V_0-\sigma_v \leq v^s \leq V_0+\sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l \leq N_t+v^s, V_0-\sigma_v \leq v^l \leq V_0+\sigma_v} (N_t - v^l + v^s) dv^l \\
&- \int_{N_t+v^s \geq 0, V_0-\sigma_v \leq v^s \leq V_0+\sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l > N_t+v^s, V_0-\sigma_v \leq v^l \leq V_0+\sigma_v} (N_t - v^l + v^s) dv^l \\
&- \int_{N_t+v^s < 0, V_0-\sigma_v \leq v^s \leq V_0+\sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{V_0-\sigma_v}^{V_0+\sigma_v} (N_t - v^l + v^s) dv^l.
\end{aligned}$$

This integral will be referred as the basic integral in the rest of the appendix. The basic integral is difficult to evaluate as it depends on the value of initial inventory position, N_t , nonlinearly. However, for large enough $|N_t|$, (when $N_t \leq -(V_0 + \sigma_v) \leq 0$ or $N_t \geq V_0 + \sigma_v$), the relationship becomes linear, as it can be proven that $E_t[|N_{t+1}|] = |N_t|$.

Range 1: $N_t \geq V_0 + \sigma_v$. As N_t is positive, this is the case that the initial inventory position is long on the security. With such a large positive inventory, $v^l \leq N_t + v^s$ is always satisfied, thus the second and third term vanish in the basic integral as neither condition $v^l > N_t + v^s$ nor $N_t + v^s < 0$ are satisfied. Therefore,

$$\begin{aligned}
E_t[|N_{t+1}|] &= \int_{N_t+v^l \geq 0, V_0-\sigma_v \leq v^s \leq V_0+\sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l \leq N_t+v^s, V_0-\sigma_v \leq v^l \leq V_0+\sigma_v} (N_t - v^l + v^s) dv^l \\
&= \int_{V_0-\sigma_v}^{V_0+\sigma_v} \frac{1}{4\sigma_v^2} \left[-\frac{1}{2} (N_t - v^l + v^s)^2 \Big|_{v^l=N_t+v^s}^{v^l=V_0+\sigma_v} \right] dv^s \\
&= \frac{1}{8\sigma_v^2} \int_{V_0-\sigma_v}^{V_0+\sigma_v} \left[(N_t + v^s - V_0 + \sigma_v)^2 - (N_t + v^s - V_0 - \sigma_v)^2 \right] dv^s \\
&= \frac{1}{24\sigma_v^2} \left[(N_t + 2\sigma_v)^3 - (N_t)^3 - (N_t)^3 + (N_t - 2\sigma_v)^3 \right] \\
&= \frac{1}{24\sigma_v^2} \left[3N_t(2\sigma_v)^2 + 3N_t(2\sigma_v)^2 \right] = N_t.
\end{aligned}$$

Range 2: $N_t \leq -(V_0 + \sigma_v)$. As N_t is negative here, this is the case that initial inventory position is short on the securities. The very large negative inventory implies that $N_t + v^s \geq 0$ cannot be satisfied with strict inequality, hence the first two terms in the basic integral vanish. Consequently,

$$\begin{aligned}
E_t[|N_{t+1}|] &= - \int_{N_t + v^s < 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{V_0 - \sigma_v}^{V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\
&= - \int_{V_0 - \sigma_v}^{V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{V_0 - \sigma_v}^{V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\
&= \frac{1}{8\sigma_v^2} \int_{V_0 - \sigma_v}^{V_0 + \sigma_v} \left[(N_t + v^s - V_0 - \sigma_v)^2 - (N_t + v^s - V_0 + \sigma_v)^2 \right] \\
&= -\frac{1}{2\sigma_v} \int_{V_0 - \sigma_v}^{V_0 + \sigma_v} (N_t + v^s - V_0) dv^s = -\frac{1}{4\sigma_v} (N_t + v^s - V_0)^2 \Big|_{v^s=V_0-\sigma_v}^{v^s=V_0+\sigma_v} \\
&= -\frac{1}{4\sigma_v} \left[(N_t + \sigma_v)^2 - (N_t - \sigma_v)^2 \right] = -N_t = |N_t|.
\end{aligned}$$

Therefore, for large initial inventory positions such that $N_t \geq |V_0 + \sigma_v|$, the equilibrium percentage spread grows linearly with the size of the inventory.

Appendix 2: More about Case 2.

In case 2 in section 4, the focus is when the initial inventory position is relatively large, at which time the expected end of period inventory grows linearly with the initial inventory. For initial inventory positions that are more moderate, the expected end of period inventory still grows with the initial inventory, but not necessarily linearly. The relationship is more complicated, and depends on the relative magnitudes of the parameters. This appendix demonstrates the nonlinear relationship with two examples.

Again, under the assumption that price and transaction volumes are independently distributed, buy and sell volumes are independent and identically distributed, and they both follow uniform distributions:

$f(v) = \frac{1}{2\sigma_v}$ in the range of $V_0 - \sigma_v \leq v \leq V_0 + \sigma_v$, and zero otherwise. As the volume cannot

be negative, it is necessary that $V_0 \geq \sigma_v$. Then (suppress the subscripts in v_{t+1}^l and v_{t+1}^s),

$$\begin{aligned}
 E_t[|N_{t+1}|] = & \int_{N_t + v^s \geq 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l \leq N_t + v^s, V_0 - \sigma_v \leq v^l \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\
 - & \int_{N_t + v^s \geq 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l > N_t + v^s, V_0 - \sigma_v \leq v^l \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\
 - & \int_{N_t + v^s < 0, V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{V_0 - \sigma_v}^{V_0 + \sigma_v} (N_t - v^l + v^s) dv^l.
 \end{aligned}$$

This integral will be referred as the basic integral in the rest of the appendix. The basic integral is difficult to evaluate as it depends on the value of initial inventory position, N_t , nonlinearly, in the range of, $V_0 - \sigma_v < N_t < V_0 + \sigma_v$. Further, in order to integrate the basic integral, the relative magnitudes of V_0 , $2\sigma_v$, and $3\sigma_v$ need to be specified. For example, if we further assume that $V_0 \geq 2\sigma_v$, then we can prove the following:

1. For $2\sigma_v \geq N_t \geq 0$, $E_t[|N_{t+1}|] \geq \frac{\sigma_v}{3} \left[\frac{1}{2} + \left(\frac{N_t}{2\sigma_v} \right)^2 \right]$.
2. For $-\sigma_v \leq N_t < 0$, $E_t[|N_{t+1}|] \geq \frac{\sigma_v}{6} \left[1 + \left(\frac{N_t}{\sigma_v} \right)^2 \right]$.

Proof:

1. $2\sigma_v \geq N_t \geq 0$: Then $N_t + V^s < 0$ will never be satisfied, thus the third term of the basic integral vanishes. As a result,

$$E_t[|N_{t+1}|] = \int_{V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{V_0 - \sigma_v \leq v^l \leq N_t + v^s, N_t + v^s \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l$$

$$- \int_{V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{N_t + v^s \leq v^l \leq V_0 + \sigma_v, N_t + v^s \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l$$

That is,

$$E_t[|N_{t+1}|] = \int_{V_0 - \sigma_v}^{V_0 + \sigma_v - N_t} \frac{1}{4\sigma_v^2} dv^s \int_{V_0 - \sigma_v}^{N_t + v^s} (N_t - v^l + v^s) dv^l - \int_{V_0 - \sigma_v}^{V_0 + \sigma_v - N_t} \frac{1}{4\sigma_v^2} dv^s \int_{N_t + v^s}^{V_0 + \sigma_v} (N_t - v^l + v^s) dv^l$$

$$= \frac{1}{4\sigma_v^2} \int_{V_0 - \sigma_v}^{V_0 + \sigma_v - N_t} \left[\frac{1}{2} (N_t + v^s - V_0 + \sigma_v)^2 + \frac{1}{2} (N_t + v^s - V_0 - \sigma_v)^2 \right] dv^s$$

$$= \frac{1}{24\sigma_v^2} [(2\sigma_v)^3 - (N_t)^3 + 0 - (N_t - 2\sigma_v)^3] = \frac{2\sigma_v}{3} - \frac{N_t}{4} + \frac{(N_t)^2}{4\sigma_v} - \frac{(N_t)^3}{12\sigma_v^2}$$

Notice that $2\sigma_v \geq N_t \geq 0 \Rightarrow 1 \geq \frac{N_t}{2\sigma_v} \geq 0$, the above expression implies

$$E_t[|N_{t+1}|] = \frac{2}{3}\sigma_v - \frac{2\sigma_v}{4} \left(\frac{N_t}{2\sigma_v} \right) + \sigma_v \left(\frac{N_t}{2\sigma_v} \right)^2 - \frac{8\sigma_v}{12} \left(\frac{N_t}{2\sigma_v} \right)^2 \left(\frac{N_t}{2\sigma_v} \right)$$

$$\geq \frac{2}{3}\sigma_v - \frac{2\sigma_v}{4} + \left(\frac{N_t}{2\sigma_v} \right)^2 \left(\sigma_v - \frac{8\sigma_v}{12} \right) = \frac{\sigma_v}{3} \left[\frac{1}{2} + \left(\frac{N_t}{2\sigma_v} \right)^2 \right]$$

2. $-\sigma_v \leq N_t < 0$: Then $N_t \geq -(V_0 - \sigma_v)$ since $V_0 \geq 2\sigma_v$, which implies that $N_t + v^s < 0$ is never satisfied. In other words, the third term of the basic integral equals to zero.

Therefore,

$$\begin{aligned}
E_t[|N_{t+1}|] &= \int_{V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l \leq N_t + v^s, V_0 - \sigma_v \leq v^l \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\
&\quad - \int_{V_0 - \sigma_v \leq v^s \leq V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{v^l > N_t + v^s, V_0 - \sigma_v \leq v^l \leq V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\
&= \int_{V_0 - \sigma_v - N_t}^{V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{V_0 - \sigma_v}^{N_t + v^s} (N_t - v^l + v^s) dv^l - \int_{V_0 - \sigma_v - N_t}^{V_0 + \sigma_v} \frac{1}{4\sigma_v^2} dv^s \int_{N_t + v^s}^{V_0 + \sigma_v} (N_t - v^l + v^s) dv^l \\
&= \int_{V_0 - \sigma_v - N_t}^{V_0 + \sigma_v} \frac{1}{8\sigma_v^2} [(N_t + v^s - V_0 + \sigma_v)^2 + (N_t + v^s - V_0 - \sigma_v)^2] dv^s \\
&= \frac{1}{24\sigma_v^2} [(N_t + 2\sigma_v)^3 + (N_t)^3 - 0 - (-2\sigma_v)^3] = \frac{2\sigma_v}{3} + \frac{N_t}{2} + \frac{N_t^2}{4\sigma_v} + \frac{N_t^3}{12\sigma_v^2}.
\end{aligned}$$

Notice that $-\sigma_v \leq N_t < 0 \Rightarrow 0 \leq \frac{N_t}{-\sigma_v} \leq 1$, therefore,

$$\begin{aligned}
E_t[|N_{t+1}|] &= \frac{2\sigma_v}{3} - \frac{\sigma_v}{2} \left(-\frac{N_t}{\sigma_v}\right) + N_t^2 \left[\frac{1}{4\sigma_v} - \frac{1}{12\sigma_v} \left(-\frac{N_t}{\sigma_v}\right)\right] \\
&\geq \frac{2\sigma_v}{3} - \frac{\sigma_v}{2} + N_t^2 \left(\frac{1}{4\sigma_v} - \frac{1}{12\sigma_v}\right) = \frac{\sigma_v}{6} \left[1 + \left(\frac{N_t}{\sigma_v}\right)^2\right].
\end{aligned}$$

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