Currency Competition:
A Partial Vindication of Hayek*

Antoine Martin and Stacey L. Schreft

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Abstract: This paper establishes the existence of equilibria for environments in which outside money is issued competitively. Such equilibria are typically believed not to exist because of a classic overissue problem: if money is valued in equilibrium, an issuer produces money until its value is driven to zero. By backward induction, money cannot have value in the first place. This paper shows that overissuance is not a problem if agents believe that if an issuer produces more than some threshold number of notes, then only those notes issued up to the threshold will be valued; additional notes will be worthless. This result is very general, applying to any monetary economy in which equilibria with and without valued money exist if the money supply is finite. The paper also compares the allocation achieved by a monopolist to that achieved with competitive issuance in both a search and an overlapping-generations environment. The results depend on the environment considered, but two general conclusions arise. First, it is ambiguous whether competitive issuers can achieve a more desirable allocation than a monopolist. Second, with competitive issuance, a licensing agency can always improve on pure laissez-faire and achieve the efficient allocation in the long run.

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*Authors' contact information: Antoine Martin, Research Department, Federal Reserve Bank of New York, 33 Liberty Street, New York, NY 10045, Antoine.Martin@ny.frb.org, 212-720-6943; Stacey Schreft, Research Department, Federal Reserve Bank of Kansas City, 925 Grand Blvd., Kansas City, MO 64198, Stacey.L.Schreft@kc.frb.org, 816-881-2581. The authors thank Warren Weber, Randy Wright, and participants in the Federal Reserve Bank of Cleveland's 2003 Banking and Payments Workshop, the 2004 SED conference in Paris, and at various university seminars for helpful comments. They also thank Aarti Singh for valuable research assistance. The views expressed are the authors' and do not necessarily reflect those of the Federal Reserve Bank of Kansas City or the Federal Reserve System.

Discussion [of my proposal to allow competition in the issuance of outside money] … cannot begin soon enough. Though its realization may be wholly impracticable so long as the public is mentally unprepared for it and uncritically accepts the dogma of the necessary government prerogative, this should no longer be allowed to act as a bar to the intellectual exploration of the fascinating theoretical problems the scheme raises. —*Friedrich Hayek*, 1990, p. 26.

1. Introduction

There is a long literature advocating the competitive issuance of fiat money—money that is intrinsically worthless and inconvertible, and thus an outside money, one that is not a liability of the issuer. Hayek is perhaps the most prominent contributor to this literature, largely thanks to his 1976 book *Denationalisation of Money*. There Hayek describes how an equilibrium with competitive issuance of outside money could come about and argues that such an equilibrium would likely dominate the equilibrium arising when the government monopolizes currency issuance.[1]

This paper evaluates Hayek's position in a formal, general equilibrium monetary model. It establishes the existence of an equilibrium with private issuance of outside money and compares the allocations obtained by monopoly and competitive issuers. There appears to be no general proof of the existence of equilibria when outside money is privately issued. Indeed, it is more commonly argued that such equilibria cannot exist. Hellwig (1985) argues this more generally, while Calvo (1978) and White (1999), among others, appeal to a time-consistency problem that affects any unregulated issuers. Proofs of nonexistence have been provided by Ritter (1995) in a search model and by Taub (1985) and Bryant (1981) in an overlapping-generations (OG) model.

Arguments justifying the nonexistence of equilibria with private issuance of outside

[1] Hayek says that as a practical matter private issuers (banks) would back their currencies with a generally accepted medium of exchange. But he was clearly thinking of such currencies as outside money. This is apparent from the following quotes: "The bank would of course not be legally liable to redeem its notes at [the promised redemption] value…," and "The outstanding notes and deposits of such a bank are not claims on it in terms of some other unit of value; it determines itself the value of the unit in terms of which it has debts and claims and keeps its books" (Hayek, 1990, p. 49, footnote 1, and p. 50, respectively).

currency typically go as follows. If issuing new money is costless and money has some positive value, then an agent with the right to issue money will do so in infinite quantity, thus inflating away the money's value. Hence, in the limit, with an infinite stock of money issued, money has no value, and by backward induction, no equilibrium exists with valued outside money. Time inconsistency is a key feature of this argument because issuers always want to believe they will constrain their note issuance, but when they need to they never have the incentive to do so.

This paper differs from most of the existing literature in focusing on the possibility that privately issued outside money may be worthless in equilibrium even when issued in finite quantities. Equilibria with valued, privately issued outside money are shown to exist if agents believe that all notes issued up to some threshold level will be valued, but additional notes will be worthless. These beliefs create a discontinuity in the value of the marginal unit of money. This discontinuity, in turn, undoes the logic of the standard nonexistence result. Because the value of a marginal unit of money reaches zero for some finite money supply, the limit argument no longer applies.

The logic behind this existence result is very general and applies to any monetary model in which money is worthless if all agents believe it has no value. To emphasize this point, this paper formally establishes existence in environments similar to the environments in which nonexistence has previously been established. It does so first in a search environment similar to that of Ritter (1995), and second in an OG environment that resembles that in Taub (1985). In each environment, the allocations under competitive and monopoly issuance are compared. The monopolist is modeled in the search environment as a coalition of otherwise unremarkable agents. This modeling approach is typical in matching models of money (e.g., Li, (1995), Ritter (1995), Aiyagari and Wallace (1997), and Li and Wright (1998)). To facilitate comparison of the two environments, the monopolist is modeled in the same way in the OG model.

The results obtained depend on the environment considered. In the search environment, competitive issuers achieve the efficient quantity of money in the long run only for specific parameterizations that have measure zero in the parameter space. A monopolist does not achieve the efficient money supply either. In the OG environment, the efficient allocation is achieved in finitely many periods if agents incur a cost of becoming money issuers. A monopoly issuer might achieve as desirable an allocation, but only if its actions are sufficiently constrained by agents' beliefs.

In taking this approach, this paper foregoes characterizing all possible equilibria for either the OG or search environments presented. There are certainly many assumptions about beliefs, and the communication needed to support those beliefs, that could be considered, and many types of equilibria could arise. Some would have privately issued money in circulation; some would not. Studying a wider range of those equilibria would be an interesting exercise in its own right, although beyond the scope of this paper. Here, the goal is simply to show that there are some assumptions on beliefs and communication that give rise to the equilibrium with competitively supplied fiat money that Hayek and others have envisioned.

Two general conclusions emerge from the analysis. First, it is ambiguous whether competitive issue should be preferred to monopoly issue. Second, when money is issued competitively, a licensing agency that sets the cost incurred by agents who choose to become money issuers can always achieve the efficient allocation in the long run. The paper thus partially vindicates Hayek. On the one hand, it is shown that equilibria with competitive issue of fiat money can exist, as Hayek argued. On the other hand, it is not clear that such equilibria have desirable welfare properties. Finally, a pure laissez-faire approach is always weakly dominated by the introduction of a regulating institution.

In showing that competitive issuance of money is feasible, this paper contributes to the literature that Hayek's writing stimulated. Klein (1974) provides an early technical argument based on reputation formation for the existence of equilibria with competitive issuance, while more recently Monnet and Berentsen have provided more formal analyses that are closer in spirit to this paper.[2] Monnet (2002) uses a deterministic-matching model to compare the allocation achieved when money is issued by private agents to the allocation when money is issued by a

[2] Hayek (1990, p. 27) developed his proposal for currency competition independently, but acknowledged his subsequent discovery that Klein's work, first presented in 1970, preceded his own.

public agent (an agent producing a public good). Although an equilibrium with valued fiat money issued by private agents exists in his model, the argument is less general since he obtains existence by assuming that agents know the exact sequence of meetings. In his framework, public money is optimal. In a random-matching model like those here, Berensten (2000) shows that there exists a time-consistent policy that allows a monopolist to supply money. Since Berentsen does not consider competitive issuance of money, he cannot do the welfare comparisons done here. Moreover, he does not consider private issuance in overlapping-generations environments, so he cannot compare across types of models, as done here.

In considering the competitive issuance of *outside* money, this paper is parallel to a growing and relatively recent literature considering the competitive private issuance of *inside* money, money that is a liability of the issuer, that the issuer promises to redeem. Cavalcanti and Wallace (1999a and b), Cavalcanti, Erosa, Temzelides (1999), and Williamson (1999) exemplify this line of research.[3] The literature on competitive *outside* money issuance is closely related to the work on competitive inside money issuance in that in some sense the real novelty in both cases is that they find equilibria with competitive issuance. In so doing, these papers expand economists' understanding of what competitive issuance might look like and how the economy would perform under it. They thus stand in sharp contrast to standard models of money with government-issued currency.

These literatures diverge, however, in that there has not been much question about the possibility of existence of privately issued inside money. Since inside money is by definition an obligation of the issuer, the promise to redeem, if credible, is enough for inside money to be valued. And in fact, valued private inside money has been around since the creation of deposits. Equilibria with outside money—money that is not an obligation of any agent in the economy—have proven more difficult to achieve theoretically and have rarely been witnessed in practice.

Another point of divergence concerns the endogeneity of the number of issuers. Here,

[3] Cavalcanti and Wallace (1999b) actually considers the case where both inside and outside money coexist. The outside money is an exogenously given stock of currency that is not a liability of any agent in the economy. The inside money is issued by private agents known as banks and is a liability of all issuers since banks in their paper are required to accept any privately issued money presented to them.

under competition, there is free entry into money issuing, subject to a one-time cost in terms of disutility. Each agent chooses whether to issue money or not, and the fraction of agents issuing is determined in equilibrium. This contrasts with the existing literature, such as Cavalcanti and Wallace (1999a and b), which takes the fraction that issue, even under competition, as exogenous.

This paper is also related to a third literature, the literature endogenizing the supply of money in search models. Contributions to that literature include Araujo and Camargo (2001), Burdett, Trejos, and Wright (2001), and Peterson (2001), as well as the paper by Camera, Craig, and Waller (2003), which focuses on competition among existing currencies.[4]

The rest of the paper is organized as follows. Section 2 presents the search environment and compares the allocations achieved by a monopolist to those achieved by competitive issuers. Section 3 undertakes the same exercise in an OG environment, and section 4 concludes.

2. A Search-Theoretic Environment

The first environment considered is a search model of money of the type developed by Kiyotaki and Wright (1991, 1993). The case with competitive money issuance is studied first, followed by the case of monopoly issuance. Welfare is then compared across each case. The private issue of outside money has already been studied in such models by Ritter (1995). Ritter finds that outside money can be valued in equilibrium if supplied by a monopolistic coalition of agents, but not if supplied competitively. This section considers a search environment similar to Ritter's. With competitive issuance and the beliefs assumed in this paper, an equilibrium with privately issued money exists whenever agents can make common knowledge the total number of notes they have issued. With a monopolistic issuer, an equilibrium with valued privately

[4] Camera, Craig, and Waller (2003) consider the circulation velocities of two currencies that have different fundamentals. Hence, they are interested in how fundamentals affect a buyer's willingness to spend a currency rather than a seller's willingness to accept notes. They do not consider currency issuance.

issued fiat money is also possible, though beliefs might not constrain a monopolistic issuer the way they do competitive issuers. As a result, equilibria with monopolistic issuance do not necessarily dominate equilibria with competitive issuance.

2.1. The Physical Environment

The economy is populated by a mass 1 of agents who live forever. Time is discrete and denoted by $t = 1, 2, \ldots$. At every date, agents meet in randomly matched pairs. There are $k > 2$ types of agents and k types of consumption goods in each period. Consumption goods are indivisible and cannot be stored.

Preferences are such that agents of type i get utility u from consuming good i and no utility from consuming any other goods. These agents can also produce good $i+1$, modulo k, at a cost c in utility, where $u > c > 0$. This guarantees that no meeting results in a double coincidence of wants and that no barter will take place.[5]

A technology exists for producing storable, nondepreciating, and nonfalsifiable objects called money that do not directly yield utility. Monies thus can be thought of as pieces of paper currency. It is assumed that the technology allows agents to differentiate their money from the money produced by others, and that agents can at all times make public knowledge the total amount of money issued to date. Both assumptions are fairly standard; models of multiple currencies typically assume that the currencies can be differentiated and that the stock of each money is known by all agents (see, for example, Matsuyama, Kiyotaki, and Matsui (1993), and Shi (1995)). One difference between those models and the one in this paper is that in this paper the money supply is not given, but determined endogenously. Hence, it should be noted that, in principle, any issuer of an outside money has an incentive to convince the public that its notes will have, and maintain, their value.[6] Further, historical evidence from the Free Banking Era in

[5] Ritter (1995) considers an environment in which barter is possible and $c = 0$. In this paper barter is ruled out for simplicity. If $c = 0$, money is not essential because gift giving is always an equilibrium (indeed, that equilibrium Pareto dominates the monetary equilibrium). Hence the focus here is on environments with $c > 0$.

[6] As Klein (1974) put it, issuers of money, like producers of any product, have an incentive to establish a favorable reputation.

the U.S. suggests that even with privately issued currency, the stock of money was fairly well known. During that period, state bank notes had serial numbers. Banks also kept records of what notes went out, to whom, and when they were redeemed. Moreover, banks' conditions were periodically reported in newspapers and to the legislature. The timing of such reports varied by city and state.[7]

Agents can become money issuers only at date $t = 1$. Becoming a money issuer and gaining access to the technology yields disutility of $\delta \geq 0$ at that initial date.[8]

Since by assumption there is no double coincidence of wants, agents must decide whether to exchange goods for money. This decision depends on agents' beliefs about other agents' willingness to exchange goods for money. This paper considers a particular type of beliefs, referred to as *L*-beliefs. Let θ_α be a public signal observed at date 1 only. θ_α takes values on the set $\{0,1,2,\ldots\}$ and is distributed according to pdf *f*. For each money α, there is a number $L_\alpha\left(\theta_\alpha\right)$ for all *t* such that all agents believe that the first L_α units of money α will be accepted in exchange for goods but that no additional units of money α will be valued. *L*-beliefs include as a special case—the case with $L = \infty$—the standard beliefs typically assumed in monetary economics, namely that any units of currency issued are valued in a monetary equilibrium. They are self-fulfilling and constrain the amount of money that competitive issuers can provide. As will be shown below, when agents have *L*-beliefs, equilibria can exist when money is issued competitively.[9] Without loss of generality, $L_\alpha = L$ for all α is assumed. Additionally, for simplicity, attention is restricted to pure strategies. Specifically, money is exchanged for goods with either probability 0 or probability 1.

Monies are subject to a storage constraint: an agent can carry at most one unit of money.

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[7] The authors are indebted to Warren Weber for providing this information.

[8] δ can be thought of as an entry fee or, alternatively, as the cost incurred by issuers to make public knowledge the number of notes they have issued, say by hiring an outside auditor to guarantee their note count, and advertising the findings.

[9] Other types of beliefs can be supported as an equilibrium as well (see, for example, Wright 1994). These beliefs are meant to capture Hayek's (1990, p. 51) view that "[An issuer] would know that the penalty for failing to fulfill the expectations raised would be the prompt loss of the business."

Allowing agents to observe the signal at additional dates could have been assumed as well. However, that significantly complicates the analysis.

With this storage constraint, along with the indivisibility of money and goods, the exchange rate between any two valued monies must be one. Allowing for divisible goods as in Trejos and Wright (1995) would complicate the exposition without modifying the results of the paper in a meaningful way. Additional equilibria could exist, with different currencies trading at different prices, as in Aiyagari, Wallace, and Wright (1996). Even if all currencies had to trade at the same price, there could be equilibria with different price levels, as in Shi (1995). Here, however, the concern is only with whether currency is issued and accepted, and the results on those dimensions should remain unchanged.

The following definition may now be stated:

Definition: An equilibrium at each *t* is a sequence of money supplies $\left\{M_t^a\right\}_{t=1}^{\infty}$ for each money α, trade decisions, and beliefs by agents such that, at each *t*, knowing M_t^α for each α and given the trade decisions and beliefs of all other agents, the trade decisions are a best response and beliefs are verified.

2.2. Competitively Supplied Money

As mentioned above, agents issue outside money; that is, notes are not a liability of the issuer. Further, as is shown in Appendix A, Claim 1, in equilibrium an issuer might not accept a note issued by another agent. More importantly, an issuer might not even accept its own notes, yet they are still valued in equilibrium.[10]

There is free entry into money issuing. A, with mass μ, denotes the set of agents α who can issue money. Given μ and the beliefs of agents about the value of money, the path of the money stock can be determined.

Agents play a two-stage game. In the first stage, they choose whether to become money

[10] This is in contrast to Cavalcanti and Wallace (1999 a and b), for example. These authors study inside money and assume issuers who do not accept a note, regardless or who issued it, are punished with autarky. Without this punishment, the authors would not have equilibria with valued privately issued notes. Thus, the fact that the notes in their models are a liability of the issuer—are inside money, not outside money—is critical for their results.

issuers. In the second stage, they are randomly matched and can trade with each other. The game can be solved by backward induction. First, the trading stage can be solved for a given value of μ. The solution reflects the value of being a money issuer and relies on agents' L-beliefs. Proposition 1, which implies that the money supply in this economy cannot exceed μL, can now be stated.

Proposition 1: A money issuer will issue at most L notes.

Proof: Consider an agent who can exchange goods for the $L+1$ note of a money issuer. This agent believes the note will be worthless if he accepts it because no other agent will exchange goods for the note. Hence, he will never give up goods for the note. ■

Given the value of issuing money from the trading stage, the first stage of the game can be solved. It will be shown that, for given parameters, only one value of μ is consistent with an equilibrium. If in equilibrium $\mu \in (0,1)$, then agents must be indifferent between issuing and not issuing money. Agents play a mixed strategy in the first stage of the game and choose to become money issuers with probability μ. It is assumed that a law of large numbers holds such that the mass of agents that become money issuers is μ and issuers are identically distributed across the k types.

It remains to determine μ as a function of δ, the disutility from becoming an issuer. Agents will become money issuers if the expected gain from doing so is at least as high as δ. For simplicity, money issuers are assumed not to produce goods in exchange for a unit of money.[11] Given this assumption, if $\delta = 0$, then all agents become money issuers and there is no equilibrium in which money is valued because no agent will accept money in exchange.[12] If $\delta > 0$, then some agents will not become money issuers. As is shown below, for a given μ,

[11] Appendix A shows that there exist parameter values for the model such that this is individually rational for money issuers. Alternatively, one could assume that the storage constraint prevents a money issuer from holding both the technology necessary for producing money and one unit of money.
[12] The case Ritter (1995) considers is equivalent to the one here with $\delta = 0$ and $L = \infty$.

there exists a $\delta > 0$ such that agents are indifferent between issuing and not issuing money. For δ sufficiently high, no agent issues money and autarky results.

The rest of this subsection characterizes equilibria. The general case with $L \geq 1$ is considered first, and the simpler case with $L = 1$, for which a closed-form solution is provided, is presented second.

Given the mass μ of potential money issuers, there is a mass $I_t(j)$ of issuers who can issue up to j units of money at date t. The money stock carried into date t, M_t, then satisfies

$$M_t = L\mu - \sum_{j=1}^{L} j\, I_t(j).$$

That is, the money stock at the beginning of period t is equal to the maximum number of notes that can be issued, less the number of notes that have not been issued yet. This implies that the mass of agents who are either holding a note or able to issue a note in period t is given by

$$\tilde{M}_t \equiv M_t + \sum_{j=1}^{L} I_t(j) = L\mu - \sum_{j=1}^{L} (j-1) I_t(j). \tag{2.1}$$

These are the agents who do not produce goods in exchange for money. It follows that $m_{0,t} \equiv \left(1-\tilde{M}_t\right)/k$ is the probability of meeting an agent who is not holding a note and cannot issue one. Likewise, $m_{1,t} \equiv \tilde{M}_t / k$ is the probability of meeting someone holding a note or able to issue one. With this notation, the law of motion for $I_t(j)$ is

$$I_{t+1}(j) = \begin{cases} I_t(L)\left[1-m_{0,t}\right] & \text{if } j = L, \\ I_t(j+1) m_{0,t} + I_t(j)\left[1-m_{0,t}\right] & \text{if } j < L, \end{cases} \tag{2.2}$$

given $I_1(L) = \mu$, and $I_1(j) = 0, j < L$, since all money issuers have the ability to issue L notes at $t = 1$. The sequences M_t and $\tilde{M}_t$ can be constructed using (2.2).

Finally, to complete the characterization of equilibria with competitive issuance when $L \geq 1$ requires determining the relationship between the mass of money issuers, μ, and the cost (disutility), δ. Intuitively, for each fixed cost of becoming a money issuer, there corresponds a fraction of the population that chooses to issue money. The greater the cost is, the smaller the fraction of the population that issues money. And the fewer people issuing money, the higher the value of being a money issuer and the lower the value of being a nonissuer. In equilibrium, the fraction of the population issuing money adjusts so that agents are indifferent between issuing and not issuing money. Of course, there are some values of δ so high that no one issues, and some so low that everyone issues and eventually there are no nonissuers to accept money in exchange. Thus, there exists an equilibrium with competitive money issuance for all δ within some interval.

The formal proof of existence requires the value function for each type of agent in the economy. $V^{\delta}_{I_t(j)}$ denotes the value function of a money issuer who can issue *j* units of money when the entry cost is δ. $V_{1,t}$ and $V_{0,t}$ denote the value functions for a nonissuer who holds one unit of money and a nonissuer who does not hold a unit of money, respectively. These value functions depend on the probability of each type of agent meeting agents with whom he could trade. An agent who does not hold money will produce goods upon meeting an agent of the correct type who either holds a unit of money or is a note issuer who has not yet issued *L* units of money. The probability of the union of these two events is m_1. An agent who holds a unit of money is in a position similar to that of a money issuer. Both can consume if they meet an agent of the right type who is not holding a note and who cannot issue one. The probability of meeting such an agent is m_0. Thus, the value functions take the following forms:

$$V^{\delta}_{I_t(j)} = m_{0,t}\left[u + \beta V^{\delta}_{I_{t+1}(j-1)}\right] + \left(1 - m_{0,t}\right)\beta V^{\delta}_{I_{t+1}(j)}, \tag{2.3}$$

$$V_{1,t} = m_{0,t}\left[u + \beta V_{0,t+1}\right] + \left(1 - m_{0,t}\right)\beta V_{1,t+1}, \tag{2.4}$$

$$V_{0,t} = m_{1,t}\left[\beta V_{1,t+1} - c\right] + \left(1 - m_{1,t}\right)\beta V_{0,t+1}, \tag{2.5}$$

where $\beta \in (0,1)$ is the discount factor. If money issuers never produce goods in exchange for money, as has been assumed, then $V^{0}_{I_1(L)} > V_{0,1}$. This is proven in Appendix A. And in equilibrium, it must be the case that $V^{\delta}_{I_1(L)} - V_{0,1} = \delta$.

The following proposition may now be stated:

Proposition 2. With competitive issuance, there is one equilibrium for each value of $\delta \in \left(\underline{\delta}, \overline{\delta}\right)$, where $\underline{\delta} < \overline{\delta}$.

Proof: Consider pairs $\left(\delta, \mu(\delta)\right)$ such that $V^{\delta}_{I_1(L)} - V^{0}_{I_1(L)} = \delta$, so agents are indifferent between issuing and not issuing money. To each of these pairs corresponds a potential equilibrium with valued fiat money provided that $L\mu(\delta)<1$ and $\mu(\delta)>0$. If the former inequality is violated, money cannot be valued because in the limit all agents hold money and no more trading can occur. In that case, money is worthless and, by backward induction, can never have value. If the latter inequality is violated, then no money is ever issued, and thus money has no value.

It is easy to see that $\mu(\delta)$ is a decreasing function of δ. Assume that $\mu(\delta)>\mu(\delta')$. A larger value of μ makes holding money less valuable, everything else being equal. Indeed, as μ increases, $m_{0,t}$ decreases and $m_{1,t}$ increases, for all *t*. This then implies that $V_{0,t}$ is increasing in μ, while $V_{1,t}$ and $V^0_{I_t(j)}$ are decreasing in μ, for all *t* and all *j*. Since in equilibrium $V^{\delta}_{I_1(L)}-V_{0,1}=\delta$ for any δ, it must be the case that $\delta<\delta'$. Further, $\mu(\delta)$ is a continuously decreasing function of δ since $m_{0,t}$ and $m_{1,t}$ are continuous functions of μ. Hence there exists $\underline{\delta}$ such that $L\mu(\delta)<1$ for all $\delta>\underline{\delta}$, and a $\overline{\delta}$ such that $\mu(\delta)>0$ for all $\delta<\overline{\delta}$. ■

As this proposition shows, monetary equilibria exist. Indeed, money is essential in this environment. In particular, agents do not have enough information about past histories to implement the gift-giving allocation. Compared to a standard search model, the only additional information agents have in this model is the number of notes issued in the past by an issuer they meet. In contrast to Cavalcanti and Wallace (1999 a and b), they do not know the trading history of the issuer. Nor do they care whether issuers have accepted their own or anyone else's notes before. They just care that agents have not issued too many notes.

The more simple case with $L=1$, for which a closed-form solution can be obtained, is now considered. In this case, $m_0=(1-\mu)/k$ and $m_1=\mu/k$. With $L = 1$, money issuers who have already issued one note are identical to nonissuers. The value functions in equations (2.3) to (2.5) thus reduce to

$$V_0=m_1\left[\beta V_1-c\right]+\left(1-m_1\right)\beta V_0, \tag{2.6}$$

$$V_1=m_0\left[u+\beta V_0\right]+\left(1-m_0\right)\beta V_1. \tag{2.7}$$

Subtracting (2.6) from (2.7) yields

$$V_1 - V_0 = \frac{m_0 u + m_1 c}{\left[1-\beta+\beta\left(m_0+m_1\right)\right]}, \tag{2.8}$$

which implies

$$V_1 = V_0 + \frac{m_0 u + m_1 c}{\left[1-\beta+\beta\left(m_0+m_1\right)\right]} > V_0. \tag{2.9}$$

Solving (2.9) and (2.6) for V_0 then gives

$$V_0 = \frac{\beta m_0 m_1 u - \left(1-\beta+\beta m_0\right) m_1 c}{\left(1-\beta\right)\left[1-\beta+\beta\left(m_0+m_1\right)\right]}. \tag{2.10}$$

It can be shown that $V_0 \geq 0$ if and only if $\beta m_0 u \geq \left(1-\beta+\beta m_0\right) c$. This last condition implies $\beta\left(V_1 - V_0\right) \geq c$.

Agents will become money issuers if $V_1 - V_0 \geq \delta$. Setting $V_1 - V_0 = \delta$, and solving for μ, taking into account the fact that $k\left(m_0 + m_1\right) = 1$, yields

$$\mu = \frac{u - \delta\left[\left(1-\beta\right)k + \beta\right]}{u - c}. \tag{2.11}$$

Thus, for a given δ, a value of $\mu(\delta)$ can be found from (2.11) that makes agents indifferent regarding money issuing. In other words, given δ, an equilibrium exists if agents choose to become money issuers with probability $\mu(\delta)$.[13]

2.3. Monopolistically Supplied Money

As mentioned previously, Ritter (1995) shows that an equilibrium with valued fiat money exists when the money is supplied by a coalition of agents acting as a monopolist. This subsection models monopoly issuance in the spirit of Ritter and considers the impact of agents' beliefs on the equilibrium. It turns out that the monopolist might not be constrained by agents' beliefs.

Agents are selected randomly from the population of all agents to be offered admission to

[13] Alternatively, one could assume that agents face different costs $\delta_i \in R_+$. In that case, the δ_i of the marginal entrant increases with μ. It is easily shown that if the distribution of δ_i is continuous there is a cutoff $\hat{\delta}$ such that all agents with $\delta_i \leq \hat{\delta}$ become money issuers and all other agents remain nonissuers.

the coalition.[14] Those chosen are distributed uniformly across the k types of agents. They can choose whether to incur the one-time disutility δ at date 1 to join the coalition and have the ability to issue money. It is assumed that $\delta \le u/k$, the expected utility of an agent from issuing money at date 1. This ensures that all agents choose to become members of the coalition if invited.[15] A is the set of agents α of mass μ who choose to join the coalition.[16]

As in the competitive case, agents have L-beliefs regarding the value of money. They believe that only the first L notes issued by the monopolist will have value. Whether such beliefs can be supported as an equilibrium, and thus whether these beliefs constrain the monopolist, depends on the extent to which members of the coalition can act collectively and on the value of c. It is assumed that members of the coalition can be identified as such although their individual identities remain anonymous. It is further assumed that their note issuance can be monitored. These assumptions allow them to act collectively to supply money, abiding by any limit the coalition sets on the fraction of agents issuing money at a particular date. It also means that they can agree to produce for each other when they meet in single-coincidence meetings. This means that the coalition might not be limited to issuing L notes if c is sufficiently small. The ability to commit is sufficient to make additional notes (beyond L) valuable for noncoalition members, even if they believe that no other noncoalition members will accept the extra notes.[17]

In what follows, the money supply that maximizes the monopoly's utility is taken to be $\hat{M}$. Given the assumptions on the coalition, gift giving will prevail in trades among coalition

[14] This assumes for simplicity, as in Ritter, that only one coalition forms.
[15] If δ is too high, only a fraction of those offered admission would join the coalition. In this case, μ would represent the fraction of selected agents that actually join the coalition.
[16] The size of the coalition, μ, is taken as given here. Other papers in the literature on private money, whether inside or outside money, also typically take the size of the coalition or banking system as given.
[17] This differs from Ritter somewhat, however, because Ritter assumes that producing goods is costless, whereas here there is a cost of producing. Thus, in this paper, to guarantee coalition members will accept each other's notes, it must be assumed that the coalition can monitor its members. Indeed, ex post a coalition member would prefer not to produce for another coalition member because of the cost of doing so. Of course, coalition members will only commit to accept each other's notes if it allows them to achieve greater welfare ex ante. It will be shown that this is the case for some parameter values. Hence, even though in equilibrium the coalition's notes may look like inside money because coalition members agree to redeem them, it is not assumed that they redeem the notes. Rather, it is shown that it is welfare-maximizing for the coalition members to agree to do so ex ante. Again, this is in contrast to Cavalcanti and Wallace (1999 a and b), and Cavalcanti, Erosa, and Temzelides (1999) because these authors are concerned with inside money while this paper is concerned with outside money.

Additionally, Ritter does not take into account that gift giving occurs among coalition members.

members. Coalition members will choose whether to accept previously issued notes held by noncoalition members if it is in their own interest. Hence, for example, if $\hat{M} \leq L$, then coalition members will accept notes only after the steady state is reached. If, instead, $\hat{M} > L$, it might be ex ante welfare improving for coalition members to accept previously issued notes held by noncoalition members once the money supply reaches *L*. This way, a coalition member minimizes the disutility *c* incurred in production if matched with a noncoalition member with money who wants their good.

The coalition chooses the steady-state money stock $\hat{M}$ and the path of the money supply to maximize the expected utility of its representative member. Since the representative coalition member is better off consuming sooner rather than later, the coalition naturally chooses to issue money as quickly as possible until the money supply reaches the desired maximum level. Thus, the coalition's choice of γ_t, the fraction of coalition agents who can issue currency at date *t*, satisfies

$$\gamma_t = \begin{cases} 1 & \text{if } t < T, \\ 0 \leq \gamma_t \leq 1 & \text{if } t = T, \\ 0 & \text{if } t > T, \end{cases}$$

for some finite date $T \geq 0$ at which the steady state is reached. This implies that the money supply evolves according to

$$M_{t+1} = M_t + \gamma_t \mu (1-\mu)\left(\frac{1-M_t}{k}\right). \tag{2.12}$$

This says that the money supply at $t + 1$ is the money supply at *t* plus the money introduced by coalition agents who can issue money and want to do so (that is, those who can replenish their money holdings after trading with a noncoalition member who does not have money but can produce the desired consumption good).

It remains to determine $\hat{M}$. As in Ritter (1996), the thought experiment is as follows. At the optimal level $\hat{M}$, the marginal utility of increasing the money supply by a small amount should be zero. Assume that the economy is at a steady state at some date *T*. An increase in the money supply increases the expected utility of coalition members since they might be able to issue an additional note and consume. The cost comes from a reduction of expected utility in the steady state due to the increase in the money supply. Formally, the welfare of a member of the coalition is evaluated, taking into account the small increase in the money supply. Then, the derivative of this expression with respect to the new level of the money supply, evaluated at the steady state, with $M_T = M_{T+1} = \hat{M}$, should be equal to zero, and yield $\hat{M}$.

First, the expected utility of a coalition member at date *T* is

$$W_T = M_T W_{1T} + \left(1 - M_T\right) W_{0T}, \tag{2.13}$$

where M_T denotes the fraction of coalition members who hold a unit of money (which, in the steady state, is equal to the fraction of all agents holding a unit of money). W_{1T} and W_{0T} denote the expected utility of coalition members from holding one note and zero notes, respectively.

It is assumed that the money supply will be increased by a very small amount ε, so that $M_{T+1} = M_T + \varepsilon$. Note that, if money is only issued in meetings between a coalition member and a noncoalition member, the economy will not be in a steady state at date *T*+1. Indeed, the fraction of noncoalition members holding money will be greater than the fraction of coalition members holding money. To keep things tractable, it is assumed that $\left(1 - \mu\right)\varepsilon$ notes are issued to noncoalition agents and $\mu\varepsilon$ are issued to coalition members. Coalition agents are better off accepting the gift of $\mu\varepsilon$ notes from other coalition members than insisting on gift-giving. Consequently, the economy is in a steady state at date *T*+1. Specifically, it is assumed that the coalition members who are allowed to issue notes to purchase goods from noncoalition members are randomly chosen, and that the $\mu\varepsilon$ notes issued to coalition members are randomly attributed to coalition members who have produced for other coalition members.

These assumptions yield only an approximation of the steady-state money supply chosen by the coalition. The approximation will underestimate the true value of $\hat{M}$ because this approach underestimates the benefits to the coalition of increasing the money supply by ε. Indeed, what the coalition would prefer to do is to distribute the $\mu\varepsilon$ notes over time to coalition members who do not hold money and are in meetings with noncoalition members who produce the right kind of good. The money would be injected so as to maintain the fraction of noncoalition member holding money equal to $\hat{M}$. In other words, the coalition could improve the distribution of the $\mu\varepsilon$ notes by giving them to agents who can consume right away rather than by giving them to random agents. Note that if coalition agents are patient (β is close to 1), then the steady-state money supply provided by this approximation is very close to the optimal steady-state money supply. Indeed, if $\beta = 1$, the derivation gives the optimal steady-state money supply. The reason is that, when coalition agents are very patient, the fact that their consumption is delayed, in the approximation, compared to what it would be under the optimal policy, does not affect welfare very much.

Proceeding with the derivation, $\hat{M}$ satisfies

$$\left.\frac{\partial W_T}{\partial M_{T+1}}\right|_{M_T = M_{T+1} = \hat{M}} = 0,$$

where W_T is from (2.13). Appendix B shows that the solution to this expression is

$$1 > \hat{M} = \frac{(u-c) + u\dfrac{1-\beta}{\beta}\dfrac{1-\mu}{\mu}}{2(u-c) + u\dfrac{1-\beta}{\beta}\dfrac{1-\mu}{\mu}} \geq \frac{1}{2}. \tag{2.14}$$

The solution for $\hat{M}$ implies that the monopolist overissues money if $\beta < 1$ and $\mu < 1$, issuing more than the socially efficient money supply of 1/2. The reason is that with the price of money fixed, coalition members can increase their consumption—and social welfare—by issuing more notes up to the point where the money supply is 1/2. Beyond that point, they can further increase their consumption to some degree before the reduction in social welfare offsets such gains. However, the monopolist never issues so much that there cannot be a monetary equilibrium.

Additionally, the solution in (2.14) holds only for $\mu < 1$. At $\mu = 1$, with all agents in the coalition, there is a nonmonetary equilibrium with gift giving among all agents.

Since agents outside the coalition can solve for $\hat{M}$ just as easily as agents within the coalition, all agents should expect that the coalition would want to issue $\hat{M}$ notes. And agents should know that at most $\hat{M}$ notes will be accepted in exchange. Hence in this case, it is easier to understand where L might come from. Most likely, L will exactly equal $\hat{M}$.

The following proposition may now be stated and proved.

Proposition 3: Suppose all agents have beliefs with threshold *L*. If *c*, the cost of production, is sufficiently small, beliefs such that $L < \hat{M}$ cannot be supported as an equilibrium and thus do not constrain the monopolist.

Proof: Assume date *t* is the first date at which the money supply exceeds *L*, and consider the problem of a noncoalition agent who must decide whether to accept a note issued by a member of the coalition. Assume further that the agent believes that no other noncoalition agents will accept this note in period *t*+1 or later. Since new notes are issued only to noncoalition agents, and noncoalition agent may not accept notes before date *t*, the fraction of coalition members holding money, denoted by $\tilde{M}_t$, is smaller than the fraction of noncoalition agent holding money. The probability at *t*+1 of the aforementioned agent meeting a coalition member who does not hold a note and sells the good the agent desires is $\mu\left(1-\tilde{M}_t\right)/k > \mu\left(1-M_t\right)/k$, where M_t is the actual money stock at date *t*. Since coalition members commit to accept notes that were issued above the threshold, then the expected utility for the noncoalition agent of accepting a note at *t* is no less than $\beta\left[\mu\left(1-M_t\right)/k\right]u > 0$. Hence, if $c < \beta\left[\mu\left(1-\hat{M}\right)/k\right]u$, the agent is better off accepting a unit of money at *t* even if $M_t > L$. By symmetry, all agents accept the monopolist's money and the hypothesized beliefs cannot be supported as an equilibrium. ■

Clearly, since coalition members can issue their own notes, they do not need to incur the cost of producing for another agent in order to get a note. For any $\mu > 0$, there exists a *c* small enough that money will be accepted even if the coalition issues more than the threshold *L*. And coalition members have an incentive to commit to accept each others' notes since such a commitment allows them to achieve a money supply greater than *L*. If $\hat{M} > L$, and *L* is sufficiently small, then members of the coalition achieve higher expected welfare if they commit to accept each other's notes (and thus achieve a long-run money supply of $\hat{M}$) than if they do not (and achieve a long-run money supply of L). When coalition members commit to accept each other's notes, their notes are essentially inside money. However, this is a result of the model, not an assumption.

3.4. Welfare Properties of Equilibria

It remains to evaluate welfare under competitive and monopoly issuance. Following standard practice in these models, welfare is taken to be the weighted sum of agents' utilities. Two questions can be asked. First, can either a monopolistic coalition or competitive issuers necessarily achieve the optimal quantity of money in the long run? Second, in general, does monopolistic issuance yield higher welfare than competitive issuance? As will be shown, the answer to the first question is simple: No. The answer to the second question depends on the parameters of the model.

In a model identical to this one but with a constant and exogenous money supply, it is well known that the optimal money stock is 1/2. The long-run quantity of money obtained with monopolistic issuance and with competitive issuance will be compared to this value.

Proposition 2 showed that for a given L, there exists an equilibrium with competitively issued money for each pair $(\delta, \mu(\delta))$. In any such equilibrium, the long-run money supply is given by $L\mu(\delta)$. Clearly there is only one value of δ such that $L\mu(\delta) = 1/2$. The farther δ is from that value, the farther the long-run money supply is from 1/2. Thus, in general, with competitive issuance, the long-run money supply is not optimal.

In contrast, with a monopolist issuer who is not constrained by agents' beliefs, the money supply as already noted is $\hat{M}$. From (2.14), it is clear that $\hat{M}$ is independent of δ and k. As the size of the coalition increases ($\mu \uparrow 1$), the money supply chosen by the coalition converges to the optimal money supply. Likewise, as $\beta \uparrow 1$, $\hat{M}$ approaches 1/2 from above. Intuitively, the monopoly balances the immediate gain from seigniorage against the future cost to its members of having to live in an economy with too much money. The more patient agents are, the greater this cost is relative to the gains from seigniorage.

Whether expected welfare is higher with a monopolistic coalition or with competitive issuers depends on the parameters of the model. Suppose, for example, that beliefs would constrain the money supply to be arbitrarily small. Then if c is small enough and β high enough, the monopoly coalition can achieve higher welfare since it is not constrained by the threshold. Moreover, for a given long-run quantity of money, the transition to the steady state involves a smaller welfare cost with a monopolist than with competitive issuance. Indeed, the monopolist reaches the steady state as quickly as is feasible, while with competitive issuance, the quantity of money tends to the steady state as t tends to infinity. However, it could be the case that beliefs would set the competitive money supply equal to or very close to 1/2, the optimal money supply. If β is close to 1, agents are very patient and the cost of the transition has very little weight. Then, from (2.14), it can be seen that there exists a μ so small that $\hat{M}$ is close to one. If μ is decreased in this manner, then by increasing β, making agents very patient, it is possible to make the cost of having $\hat{M}$ close to one so high that welfare is higher with competitive issuance than with the monopoly.

Two other points deserve to be noted. First, it should be clear that even if the long-run money supply were equal to 1/2, the transition to the steady state would involve a welfare loss compared to the case where the money supply is given exogenously at the initial date and 1/2. Second, the fact that each money issuer incurs disutility of δ also results in a welfare loss.

Instead of taking δ as given, it could be assumed that there exists some institution in charge of licensing money issuers. This institution could choose the disutility δ from obtaining a license.[18] With competition, such a licensing institution is always able to achieve the optimal long-run quantity of money. It simply has to choose δ so that $L\mu(\delta)=1/2$. In contrast, with monopoly issuance a licensing institution is ineffective since the long-run money supply is independent of δ.

These results can be summarized in the following proposition.

[18] Under the assumption that δ is an entry cost, say a cost of acquiring the issuing technology and ability to make public knowledge the number of notes issued, the licensing agency can impose a licensing fee if the optimal cost exceeds δ or subsidize entry into note issuing if the desired cost is lower than δ.

Proposition 4: 1) Depending on the values of the parameters, competitive issue might or might not yield higher welfare than monopolistic issue. 2) In general, neither competitive issuers nor a monopolistic coalition produces the long-run optimal quantity of money. 3) Competitive issuers regulated by a licensing agency can issue the long-run optimal quantity of money.

In summary, these results partially vindicate Hayek. On the one hand, equilibria exist when money is issued competitively, as Hayek argued. However, there is no guarantee that these equilibria have desirable welfare properties. Such equilibria will achieve the optimal quantity of money only for a particular value of δ. One way to ensure that the optimal δ obtains is to assume that some institution, perhaps some government regulatory agency, charges an appropriate licensing, or entry, fee. The operation of such an institution is inconsistent with Hayek's advocacy of laissez-faire money issuance.

3. An Overlapping-Generations Environment

Taub (1985) has shown in an OG model the nonexistence of an equilibrium with competitive issuance of outside money. This section shows, in a similar OG model, that such an equilibrium can exist when agents have *L*-beliefs. This is because these beliefs eliminate the time inconsistency problem at the root of the nonexistence result in Taub's paper. The approach closely parallels the search-theoretic model of Section 2 to facilitate comparison and to emphasize that the existence result is not an artifact of the search environment.

3.1. The Physical Environment

The economy exists at dates $t = 1, 2, \ldots$. At each date, a new generation of mass 1 of two-period-lived agents is born into the economy. At date $t = 1$, the economy also consists of a mass 1 of old agents.

A single perishable and nonstorable good is available in the economy at each date. Each agent is endowed with a quantity ω of this good when young and nothing when old. Agents born in period t have preferences that can be represented by the utility function

$$U(t) = u(c_t(t)) + \beta u(c_t(t+1)),$$

where $c_t(\tau)$ denotes the period-τ consumption of an agent born at date t; the utility function u has the properties $u_i > 0$, and $u_{ii} < 0$, $i = 1, 2$; and $c_t(t)$ and $c_t(t+1)$ are assumed to be normal goods. The initial-old agents have utility function $u(c_0(1))$, where u takes the same form as for later generations.

The economy also has a technology for producing storable, nondepreciating goods, called monies, that are recognizable by all agents and nonfalsifiable. These monies are all divisible.[19] Agents must pay a resource cost of $\delta \geq 0$ to become money issuers and make their stock of money public knowledge.

3.2. Competitively Supplied Money

The monies are issued at date t by a set A_t of agents with typical member α and mass μ_t. Agents have L-beliefs regarding the value of these monies. That is, let θ_α be a public signal observed at all dates but whose value is realized only at date 1.[20] θ_α takes values on the set {0,1,2,…} and is distributed according to pdf *f*. For each money α, there is a number $L_\alpha(\theta_\alpha)$ for all t such that all agents believe that the first L_α units of money α will be accepted in exchange for goods but that no additional units of money α will be valued.[21] For simplicity, $L_\alpha = L$ for all α is assumed. Clearly, with such beliefs money issuers will not be able to issue

[19] This assumption of divisible monies contrasts with the assumption of indivisible monies in the search model. Divisibility is important here because it allows each agent who becomes a money issuer to have an equal share of the money supply and thus for there to be a representative issuer.

[20] If θ_α takes values in a bounded set, the model's steady state would not be qualitatively different if a different signal were observed at each date. It would just take longer to reach the steady state.

[21] One could assume, alternatively, that the threshold is a percentage of the money supply outstanding rather than a level. This would not modify the results regarding the existence of an equilibrium with valued fiat money. It would, however, modify the results regarding welfare. When the threshold is a percentage, the equilibrium allocation never achieves, or even converges to, the efficient allocation.

more than L notes (see proposition 1).

In this economy, the exchange rate is indeterminate as in Kareken and Wallace (1981). Assuming that all notes have the same face value and are exchanged one-for-one simplifies the exposition without altering the qualitative results. Thus, each note issued is assumed to have a face value of \$1.

The initial old generation will consume nothing unless they pay δ and issue money themselves. Since they prefer to consume, and the utility function is strictly increasing, they will issue L units of money, the most that will be valued in trade. The money supply at date 1 is thus $M_1 = L$. Agents can hold as many units of money as they choose each period.[22]

Agents in this economy solve the following problem[23]

$$\max \quad u\left(c_t^j(t)\right) + \beta u\left(c_t^j(t+1)\right), \tag{3.1}$$

subject to

$$M_t^j + p_t c_t^j(t) \le p_t\left(\omega - \chi_j \delta\right), \tag{3.2}$$

$$p_{t+1} c_t^j(t+1) \le M_t^j + \chi_j L, \tag{3.3}$$

where $j = i$ if the agent is a money issuer, $j = ni$ for nonissuers, M_t^j denotes the amount of money acquired by the agents of type j when young, p_t denotes the price of goods in terms of money in period t, and χ is an indicator function that takes value 1 if $j = i$ and 0 if $j = ni$. The utility function u has the following properties: $u_i > 0$, $u_{ii} < 0$, i = 1,2, and it is assumed that both $c_t(t)$ and $c_t(t+1)$ are normal goods. Equations (3.2) and (3.3) can be combined to obtain the intertemporal budget constraint:

$$c_t^j(t) + \frac{p_{t+1}}{p_t} c_t^j(t+1) \le \omega + \chi_j\left(\frac{L}{p_t} - \delta\right). \tag{3.4}$$

Clearly, $U^i(t) > U^{ni}(t)$ if and only if $L > \delta p_t$. Further, since the fraction μ_t of agents who become money issuers at date t satisfies

[22] This assumption is standard in OG models and is also assumed under monopoly issuance. It differs from the search model of section 2, which has a storage constraint.

[23] Agents have no incentive to produce money until they are old. The agent's problem is not significantly different if the resource cost is paid when young, as is assumed in the budget constraints, or when old.

$$\mu_t \begin{cases} = 0 & \text{if } U^i(t) < U^{ni}(t), \\ \in (0,1) & \text{if } U^i(t) = U^{ni}(t), \\ = 1 & \text{if } U^i(t) > U^{ni}(t), \end{cases} \tag{3.5}$$

equation (3.4) implies that all agents belonging to the same generation have the same consumption bundle. This is obvious if either all agents or no agents become money issuers. If $\mu_t \in (0,1)$, then it must be the case that $L = \delta p_t$, and equation (3.4) implies that all agents face the same budget constraint.[24]

It is important to notice that $U^i(t) = U^{ni}(t)$, which obtains when $L = \delta p_t$, does not imply that μ_t can take any value in $(0,1)$. For a given δ, only one value of μ_t is consistent with $U^i(t) = U^{ni}(t)$. This can be seen by recognizing that, as will be shown below, p_t depends on μ_t, so that $L = \delta p_t$ can only hold for a specific value of μ_t.

The money supply at date *t*, M_t, evolves according to $M_t = M_{t-1} + \mu_t L$ $= \mu_t\left(M_{t-1}^i + L\right) + (1-\mu_t)M_{t-1}^{ni}$. The price level adjusts to clear the money market:

$$p_t\left[\mu_t\left(\omega - c_t^i(t) - \delta\right) + (1-\mu_t)\left(\omega - c_t^{ni}(t)\right)\right] = M_t. \tag{3.6}$$

The following lemma will prove useful.

Lemma: An increase in μ_t leads to an increase in p_t.

Proof: Consider equilibrium sequences {μ_t, p_t, M_t, $c_t(t)$, $c_t(t+1)$}. Now consider an alternative sequence $\bar{\mu}_t$ for μ_t such that at date τ, $\bar{\mu}_\tau > \mu_\tau$, but μ_t and $\bar{\mu}_t$ are the same at all other dates. Assume, to establish a contradiction, that $\bar{p}_\tau \le p_\tau$. Since M_τ is given in equation (3.6), it follows that $\bar{c}_\tau(\tau) < c_\tau(\tau)$. Note that equation (3.6) can be rewritten as $p_t c_t(t+1) = M_t$, which implies that $\bar{c}_\tau(\tau+1) \ge c_\tau(\tau+1)$. Since $c_t(t)$ and $c_t(t+1)$ are both normal goods, $\bar{c}_\tau(\tau) < c_\tau(\tau)$ and $\bar{c}_\tau(\tau+1) \ge c_\tau(\tau+1)$ can only occur if $\bar{p}_{\tau+1}/\bar{p}_\tau < p_{\tau+1}/p_\tau$. And of course, $\bar{\mu}_\tau > \mu_\tau$ implies $\bar{M}_{\tau+1} > M_{\tau+1}$.

If the sequence $\bar{p}_t / p_t$, $t > \tau$, does not converge, then the sequence $c_t(t+1)$, $t > \tau$, does not converge either, which implies that feasibility is eventually violated. If, instead, the

[24] This compares to the search-theoretic environment, where issuers and nonissuers consume different bundles.

sequence $\bar{p}_t / p_t$, $t > \tau$, does converge, then this implies that $\bar{p}_{t+1}/\bar{p}_t \to p_{t+1}/p_t$, as $t \to \infty$. But then this means that $\bar{c}_t(t+1) \to c_t(t+1)$, as $t \to \infty$, which is not possible since, for all $t > \tau$, $\bar{M}_t > M_t$ and $\bar{p}_t < p_t$. ■

The long-run money supply can be shown to depend on δ. If δ is high enough, only the initial old generation issues money and $M_t = L$ for all *t*. As δ decreases, more generations issue money and the money stock grows. This is described formally in the following proposition. [25]

Proposition 5: For all $\delta > 0$, there exist $T_\delta < \infty$ such that the following are true: If $t < T_\delta$, then $\mu_t = 1$. If $t > T_\delta$, then $\mu_t = 0$. If $t = T_\delta$, then $\mu_t \in (0,1)$.

Proof: If δ is too high, it might be the case that no agents issue notes. In such a situation, $T_\delta = 0$. For the remainder of the proof, it is assumed that δ is not too high.

It is first shown that $L > \delta p_t$ cannot hold for all *t*. Assume, to establish a contradiction, that $L > \delta p_t$ for all *t*. Note that equation (3.6) can be rewritten as $p_t c_t(t+1) = M_t$, where $c_t(t+1) = c_t^i(t+1) = c_t^{ni}(t+1)$. Since $L > \delta p_t$ for all *t*, all agents of each generation become money issuers and $M_{t+1} = M_t + L$ for all *t*. Consequently, $M_t \to \infty$ as $t \to \infty$. By assumption, $p_t < L/\delta$, which implies that $c_t(t+1) \to \infty$ for the equation $p_t c_t(t+1) = M_t$ to hold. This, however, violates feasibility. Hence, there must be a finite *t* such that if all agents issue notes at that date, then $L \le \delta p_{t+1}$. Let $T_\delta \equiv t+1$. At T_δ, not all agents become money issuers, since otherwise it would be the case that $L < \delta p_{T_\delta}$. From the lemma it is evident that p_t increases with an increase in μ_t. By continuity, there must be one value of $\mu_t \in (0,1)$ such that $L = \delta p_{T_\delta}$. Finally, since $p_{T_\delta+1} > p_{T_\delta}$, $L < \delta p_t$ for all $t > T_\delta$, and no agents choose to issue notes. Since the money supply does not increase further, the economy reaches a steady state. ■

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[25] In Bryant (1981), the existence and efficiency of an equilibrium with privately issued currency (there, inside money) also depends critically on the cost of producing currency.

3.3. Monopolistically Supplied Money

To facilitate comparison to the search model, a monopoly issuer is taken to be a coalition of otherwise unremarkable agents. In each generation a set of agents of mass μ_t is selected randomly and invited to join the coalition. For simplicity, it is assumed that $\mu_t = \mu$ for all *t*. To join, these agents must pay a utility cost δ. In equilibrium, a fraction γ_t of selected agents chooses to join the coalition.[26] The coalition decides how much money to issue at each date and divides what it issues equally among its members. [27] Consequently, the agent's optimization problem with monopoly issuance is as specified in equations (3.1) through (3.3), but with (3.3) as $p_{t+1}c_t^j\left(t+1\right) \le M_t^j + \chi_j\left(L/\mu\gamma_t\right)$.

The monopoly always issues as many notes as it can. This is due to the fact that, having finite lives, the coalition's members do not benefit from moderating their money issuance. Thus the monopoly here behaves differently than it does in the search environment, where the ability to act collectively allows it to moderate its money issuance.

L-beliefs can work in one of two ways in this environment. One possibility is that the limit *L* applies to the total notes issued by the coalition for all time. In this case, the members of the initial old generation will issue the entire money supply, so $M_t = L$ for all $t \ge 1$. For the members of all future generations, belonging to the coalition confers no benefits so no one joins.

The other possibility is that the limit *L* applies to money issuance at each date. This in effect means that every generation faces a limit of *L* on the notes it can issue. As in the case of competitive issuance, each coalition member's share of the proceeds from money issuing is $L/\left(\mu\gamma_t p_t\right)$, and agents join the coalition if $L/\left(\mu\gamma_t p_t\right) \ge \delta$. If δ is sufficiently small, all agents of early generations join the coalition and issue money, and each generation issues *L* notes. The

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[26] In the OG environment, γ_t has a slightly different interpretation than it did in the search environment. In the latter case, the monopolistic coalition is chosen once and for all at the beginning of time, and γ_t denotes the fraction of coalition members allowed to issue a unit of money at date *t*. In contrast, in the OG environment, a new coalition is formed at each date, and γ_t is the fraction of those offered membership who actually join at date *t*. Agents would not pay δ to join unless they expected to issue money.

[27] This approach differs from the traditional approach to modeling the government in an OG environment. The traditional approach takes the government as funding some exogenous consumption stream through money, bond, or tax finance.

money supply thus evolves according to $M_t = M_{t-1} + L$, and the price level at date *t* is determined by a variant of (3.6):

$$p_t\left[\mu\gamma_t\left(\omega - c_t^i(t) - \delta\right) - \left(1-\mu\gamma_t\right)\left(\omega - c_t^{ni}(t)\right)\right] = M_t. \tag{3.7}$$

As the money supply increases, so does the price level, reducing the benefits from becoming a money issuer. At some date *T*, only a fraction of the selected members will agree to become money issuers (i.e., $0 < \gamma_t < 1$). Members of generation *T* will be indifferent between issuing and not issuing money. No member of any subsequent generation will become a money issuer (i.e., $\gamma_t = 0$). This can be summarized in the following proposition.

Proposition 6: For each $\delta > 0$, there exists a $T_\delta < \infty$ such that the following are true: If $t < T_\delta$, then $\gamma_t = 1$. If $t > T_\delta$, then $\gamma_t = 0$. If $t = T_\delta$, then $\gamma_t \in (0,1)$.

Proof: The proof follows the same steps as the proof of Proposition 5 and is therefore omitted. ■

3.4. Welfare Properties of Equilibria

Following Green (1997), welfare is evaluated relative to an efficient stationary allocation. This criterion has the advantage of being technically simple, as well as implying the standard Pareto-efficiency criterion. Indeed, Okuno and Zilcha (1980) show that an efficient stationary allocation is Pareto efficient in the set of all feasible allocations of the infinite-horizon economy.

An efficient stationary allocation solves the problem of maximizing the utility of the members in each age cohort. Thus, an allocation $\left(c_t^*(t), c_t^*(t+1)\right)$ is efficient if it solves

$$\max \quad u\left(c_t(t)\right) + \beta u\left(c_t(t+1)\right),$$

subject to

$$c_t(t) + c_t(t+1) \le \omega.$$

A necessary and sufficient condition for an allocation to be efficient is then

$$u'\left(c_t(t)\right) = \beta u'\left(c_t(t+1)\right), \tag{3.8}$$

which implies that $c_t^i(\tau) = c_t^{ni}(\tau)$ must hold for $\tau = t, t+1$ and for all *t*. The equilibrium allocation is efficient if $p_{t+1}/p_t = 1$ for all *t*, which comes about when no one issues money because the return from doing so is less than δ.

With competitive issuance and $\delta = 0$, all members of each generation choose to become money issuers. However, as the money stock increases, the money issued by new generations represents an ever smaller fraction of the total. So while the price is steadily rising, it increases at a decreasing rate, the inflation rate converges to 1. Thus, the equilibrium allocation converges to the efficient allocation as *t* tends to infinity.

Alternatively, from Proposition 5, if $\delta \in (\delta_{t+1}, \delta_t)$, then no one in generations after date $T_\delta + 1$ becomes a money issuer. Thus, the efficient allocation is reached after finitely many periods. The number of periods needed to reach the efficient allocation decreases as δ increases. If δ is big enough, only members of the initial old generation issue money, and the equilibrium is efficient. However, if δ is too big, then no money will be issued.

A monopolistic coalition also achieves the efficient allocation after finitely many periods. If the threshold *L* is a single limit on all issuance by the coalition across time, then the initial old generation issues *L* notes and no other generations issue notes. In this case, the efficient allocation is achieved for all generations $t \geq 1$. However, $c_0^i(1) > c_0^{ni}(1)$, which is not efficient. If instead the threshold applies only to the members of the coalition in a given generation, then the money supply grows continuously according to $M_t = M_{t-1} + L$ until some generation $T \geq 1$. No money is issued by generations $t > T$, and the allocation is efficient for these generations. In generations $t < T$, issuers consume strictly more than nonissuers.

With monopoly issuance, the efficient allocation is not only reached, but is reached in the same number of periods as with competition. Indeed, the money supply grows at the same rate with the coalition as it does with competitors—it increases by *L* every period—and all agents pay the same cost for the opportunity to issue notes. Further, until the efficient allocation is achieved, competition yields greater welfare since the monopolistic coalition introduces randomness in the consumption of ex ante identical agents. These results can be summarized in the following proposition.

Proposition 7: Both a monopolistic coalition and competitive issuers achieve the efficient allocation after finitely many periods.

Compared to the search environment, the welfare results in the overlapping generations environment are much more supportive of Hayek. First, an equilibrium with competitive issue of outside money exists. Second, for any $\delta \geq 0$, the equilibrium allocation at least converges to the efficient allocation in the long run. For $\delta = 0$, the efficient allocation is reached as $t \to \infty$. For $\delta > 0$, the efficient allocation is reached in finitely many periods, and if δ is sufficiently high, it is achieved for all $t \geq 1$. This also implies that, as was the case in the search environment, a licensing agency is able to achieve efficiency by setting δ. Here, however, it can do so under both monopolistic and competitive issuance. Of course, if the monopoly is the government, it might not choose to license or otherwise constrain itself.

4. Conclusion

This paper established the existence of an equilibrium with competitively issued outside money in both a search and an OG framework. These environments were chosen for two reasons. First, the search and OG models are arguably the most frequently used models in monetary economics. Second, for each of these frameworks it has previously been proven that a monetary equilibrium with competitive issuance cannot exist.

The existence result obtained here is expected to hold in other environments as well. Indeed, it only depends on the existence of an equilibrium where money is valued because agents believe that it will be, as well as an equilibrium where money is not valued because agents believe it will not be. This appears to apply to any model in which an equilibrium with valued fiat money can obtain.

The welfare results, however, are specific to the environment considered. Two conclusions can be drawn that apply to each environment studied. First, it is ambiguous whether welfare is higher with competitive issuance than with monopoly issuance. Second, a licensing agency can achieve the efficient allocation if money is issued competitively, at least in the long run.

If money is issued competitively in the search environment, the efficient quantity of money is in general not achieved. In the search environment, there is only one money stock consistent with optimality, and there is no guarantee that the money stock that can come about given agents' beliefs will be the optimal one. Only for a specific cost of becoming a money issuer (δ in the model) is the efficient money supply achieved in the long run. In contrast, in the OG setting, the equilibrium allocation at least converges to the efficient allocation in the long run. This occurs because agents' beliefs constrain issuers, who will always issue as many notes as will be valued.

The welfare results also differ across the two environments studied when there is monopolistic issuance. In the search environment, the monopolist is not constrained by agents' beliefs because its members can commit to accept their own currency. However, it can limit its own issuance if it is not too large and can constrain the behavior of its members. Whether monopolistic issuance dominates competitive issuance in a search environment depends on the parameters of the model, in particular on agents' beliefs (how much money they will value) and on the cost of becoming a money issuer. For certain parameter values, the coalition's size allows it to not be constrained by agent's beliefs. In such cases, the best stationary allocation achievable with competitive issuance dominates the outcome with monopolistic issuance.

In the OG environment, in contrast, the monopolistic coalition is unable to limit its issuance of money, but it is constrained by agents' beliefs in the same way that competitive issuers are constrained. Because agents have short lives, their benefit from issuing money today always exceeds the benefit from moderating issuance. Their short lives also prevent them from committing to accept coalition money in future exchanges. Thus, as in the search environment, whether monopolistic or competitive issuance is more desirable is ambiguous in the OG environment. Which dominates depends on whether agents' beliefs impose a limit on the amount issued by each generation of agents in the monopoly coalition, or if it imposes a single limit on the coalition's issuance across all periods. In the former case, competitors achieve a better allocation. In the latter case, the monopolist achieves the efficient allocation, and competitors cannot do better.

The paper also finds that in both environments a licensing agency can always do better than pure laissez-faire if it can set the cost of becoming a money issuer. In fact, it can achieve the optimal quantity of money, at least in the long run. Of course, without the ability to vary the cost, there is no role for such an agency.

It is tempting to think of a government as being able to achieve the optimal quantity of money by selling rights to issue a certain quantity of money and retaining the right to print the money, presumably of just one type, itself. But that scheme begs the question, considered elsewhere for financial intermediaries, of who monitors the monitor (Krasa and Villamil, 1992). The whole concern of Hayek and others who advocate laissez-faire in money issuance is that any government has an incentive to overissue, and historically has done so.

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Appendix A

Claim 1: There exist parameter values such that a money issuer does not produce goods in exchange for a unit of money.

Proof: Consider first the case with $L=1$. Let V_0 denote the value function of nonissuers who do not hold a unit of money. V_1 denotes the value function of a nonissuer who holds a unit of money. $V_{I_t(1)}$ denotes the value function of a money issuer at *t* who does not hold a unit of money when $L=1$. $V^+_{I_t(1)}$ denotes the value function of a money issuer who holds a unit of money when $L=1$. These value functions can be written as follows:

$$V_0 = m_1\left[\beta V_1 - \beta V_0 - c\right] + \beta V_0,$$

$$V_1 = m_0\left[u + \beta V_0 - \beta V_1\right] + \beta V_1,$$

$$V_{I_t(1)} = m_0\left[u + \beta V_0 - \beta V_{I_t(1)}\right] + m_1\left[\beta V^+_{I_t(1)} - \beta V_{I_t(1)} - c\right] + \beta V_{I_t(1)},$$

$$V^+_{I_t(1)} = m_0\left[u + \beta V_{I_t(1)} - \beta V^+_{I_t(1)}\right] + \beta V^+_{I_t(1)}.$$

The expression for $V^+_{I_t(1)}$ assumes that if the agent trades, he uses the note he is holding instead of issuing a new note. If the parameters of the model (μ, *u*, *c*, β, *k*) are such that $\beta\left(V^+_{I_t(1)} - V_{I_t(1)}\right) = c$, then this assumption is made without loss of generality because a money issuer is indifferent between accepting a unit of money and not accepting it, and so $V_{I_t(1)} = V_1$. In this case, the expressions for V_0 and V_1 are given by equations (2.10) and (2.9).

It can easily be seen that

$$\left(V^+_{I_t(1)} - V_1\right) = \frac{\beta m_0}{1-\beta+\beta m_0}\left(V_1 - V_0\right).$$

Hence, if $\beta < 1$, then $\left(V^+_{I_t(1)} - V_1\right) < \left(V_1 - V_0\right)$ for all $\mu \in [0,1]$. Since by assumption $\beta\left(V^+_{I_t(1)} - V_1\right) = c$, then by continuity there exists $\overline{\beta} < \beta$ such that holding all other parameter values fixed, $\left(V^+_{I_t(1)} - V_1\right) < \overline{\beta} < \left(V_1 - V_0\right)$. For these parameters the money issuers never agree to produce goods in exchange for money.

Since it has been established that the marginal utility of money is decreasing, it must be the case that if money issuers do not produce when $L = 1$, then they will not produce for $L > 1$. ■

Claim 2: If money issuers never produce goods in exchange for a unit of money, then $V^0_{I_t(L)} > V_0$.

Proof: Recall that

$$V_{1,t} = m_{0,t}\left[u + \beta V_{0,t+1}\right] + \left(1 - m_{0,t}\right)\beta V_{1,t+1},$$
$$V_{0,t} = m_{1,t}\left[\beta V_{1,t+1} - c\right] + \left(1 - m_{1,t}\right)\beta V_{0,t+1}.$$

Thus it is possible to write

$$V_{1,t} - V_{0,t} = \beta\left(1 - m_{0,t} - m_{1,t}\right)\left(V_{1,t+1} - V_{0,t+1}\right) + m_{0,t}u + m_{1,t}c,$$

so that $V_{1,t} > V_{0,t}$ if $V_{1,t+1} > V_{0,t+1}$. The value functions in the steady state, denoted $V_{0,\infty}$ and $V_{1,\infty}$, are given by equations (2.10) and (2.9), respectively. Since $V_{1,\infty} > V_{0,\infty}$, then $V_{1,t} > V_{0,t}, \forall t$.

Because money issuers never produce in exchange for money, $V_{1,t} = V^0_{I_t(1)}$. The value function $V^0_{I_t(2)}$ can be written as

$$V^0_{I_t(2)} = m_{0,t}\left[u + \beta V^0_{I_t(1)}\right] + \left(1 - m_{0,t}\right)\beta V^0_{I_t(2)}.$$

Thus,

$$V^0_{I_t(2)} - V^0_{I_t(1)} = m_{0,t}\beta\left[V^0_{I_{t+1}(1)} - V_{0,t+1}\right] + \left(1 - m_{0,t}\right)\beta\left[V^0_{I_{t+1}(2)} - V^0_{I_{t+1}(1)}\right].$$

Substituting iteratively into this expression for $V^0_{I_{t+i}(2)} - V^0_{I_{t+i}(1)}, i = 1, 2, \ldots$, yields

$$V^0_{I_t(2)} - V^0_{I_t(1)} = m_{0,t}\beta\left[V^0_{I_{t+1}(1)} - V_{0,t+1}\right]$$
$$+ \sum_{j=1}^{\infty} m_{0,t+j}\beta^{j+1}\left(V^0_{I_{t+j+1}(1)} - V_{0,t+j+1}\right)\prod_{i=1}^{j}\left(1 - m_{0,t+i-1}\right).$$

Since $V^0_{I_t(1)} > V_{0,t}, \forall t$, then $V^0_{I_t(2)} > V^0_{I_t(1)}, \forall t$. A similar process show that $V^0_{I_t(j+1)} > V^0_{I_t(j)}, \forall t, j$, so that $V^0_{I_t(L)} > V_0$. ■

Appendix B

This appendix solves

$$\left.\frac{\partial W_T}{\partial M_{T+1}}\right|_{M_T=M_{T+1}=\hat{M}}=0$$

for $\hat{M}$, where

$$W_T=M_T W_{1T}+\left(1-M_T\right)W_{0T}. \tag{A.1}$$

The expression for W_{1T} can be written

$$W_{1T}=\frac{\mu}{k}\left(u-c\right)+\left(1-\mu\right)\frac{\left(1-M_T\right)}{k}\left(\beta V_{0T+1}+u\right)+\left(1-\left(1-\mu\right)\frac{\left(1-M_T\right)}{k}\right)\beta V_{1T+1}.$$

The first term corresponds to the case where a coalition member meets another coalition member (probability μ) and a single coincidence of wants occurs (probability $1/k$). Coalition members practice gift giving with each other so their expected utility from such a meeting is $u-c$. The second term corresponds to the case where a coalition member meets a noncoalition member (probability $1-\mu$), who is not holding a note (probability $1-M_T$), and who produces the good the coalition member wants to consume (probability $1/k$). In that case, the coalition member consumes and starts the next period without a note. Finally, the last term corresponds to all meetings without a single coincidence of wants.

The expression for W_{0T} is

$$W_{0T}=\frac{\mu}{k}\left(u-c\right)+\left(1-\mu\right)\frac{M_T}{k}\left(\beta V_{1T+1}-c\right)+\left(1-\mu\right)\frac{\left(1-M_T\right)}{k}\frac{\left(1-\mu\right)\varepsilon}{\mu\left(1-M_T\right)}\left(\beta V_{0T+1}+u\right)$$
$$+\frac{\mu\varepsilon}{\mu\left(1-M_T\right)}\left(\beta V_{1T+1}-\beta V_{0T+1}\right)+\left(1-\left(1-\mu\right)\frac{M_T}{k}-\left(1-\mu\right)\frac{\left(1-M_T\right)}{k}\frac{\left(1-\mu\right)\varepsilon}{\mu\left(1-M_T\right)}\right)\beta V_{0T+1}.$$

The first term is the same as in W_{1T}. The second term corresponds to the case where the coalition member meets a noncoalition member (probability $1-\mu$), who is holding a note (probability M_T), and who wants to consume the good the coalition member produces (probability $1/k$). Since, by assumption, the steady state is reached at date $T+1$, coalition members cannot expect to be able issue a note and consume at any date in the future. Hence, they choose to produce in exchange for a note. The third term corresponds to the case where the coalition member meets a noncoalition member (probability $1-\mu$), who is not holding a note (probability $1-M_T$), and who produces the good the coalition member wants to consume (probability $1/k$). In this case, the coalition member may be allowed to issue a note. Since an amount $(1-\mu)\varepsilon$ of notes is going to be issued to noncoalition members, the probability that a coalition member will be able to issue a note is $\left[(1-\mu)\varepsilon\right]/\left[\mu(1-M_T)\right]$. The next term corresponds to notes that are issued to coalition members. $\mu\varepsilon$ notes are issued to coalition members who do not already hold a note. The last term corresponds to single coincidence meetings.

It remains to describe the expressions for V_{1T+1} and V_{0T+1}. These are simply the steady-state value functions for coalition members corresponding to holding a note and holding zero notes, respectively. The expressions are

$$V_{0T+1} = V_0 = \frac{\mu}{k}(u-c) + (1-\mu)\frac{M_{T+1}}{k}\left[\beta V_1 - c\right] + \left(1-(1-\mu)\frac{M_{T+1}}{k}\right)\beta V_0, \tag{A.2}$$

$$V_{1T+1} = V_1 = \frac{\mu}{k}(u-c) + (1-\mu)\frac{(1-M_{T+1})}{k}\left[\beta V_0 + u\right] + \left(1-(1-\mu)\frac{(1-M_{T+1})}{k}\right)\beta V_1. \tag{A.3}$$

Writing W_T in (A.1) explicitly, letting $\varepsilon_t = M_{T+1} - M_T$, yields:

$$\begin{aligned} W_T = {} & M_T\frac{\mu}{k}(u-c) + (1-\mu)\frac{M_T(1-M_T)}{k}(\beta V_{0T+1}+u) + M_T\left(1-(1-\mu)\frac{(1-M_T)}{k}\right)\beta V_{1T+1} \\ & + (1-M_T)\frac{\mu}{k}(u-c) + (1-\mu)\frac{(1-M_T)M_T}{k}(\beta V_{1T+1}-c) + (1-\mu)^2\frac{(1-M_T)}{k}\frac{(M_{T+1}-M_T)}{\mu}(\beta V_{0T+1}+u) \\ & + (M_{T+1}-M_T)(\beta V_{1T+1}-\beta V_{0T+1}) + (1-M_T)\left(1-(1-\mu)\frac{M_T}{k} - (1-\mu)^2\frac{(1-M_T)}{k}\frac{(M_{T+1}-M_T)}{\mu(1-M_T)}\right)\beta V_{0T+1}. \end{aligned}$$

After simplifying, this becomes

$$\begin{aligned} W_T = {} & \frac{\mu}{k}(u-c) + (1-\mu)\frac{M_T(1-M_T)}{k}(u-c) + (1-\mu)^2\frac{(1-M_T)}{k}\frac{(M_{T+1}-M_T)}{\mu}u \\ & + (1-M_{T+1})\beta V_{0T+1} + M_{T+1}\beta V_{1T+1}. \end{aligned} \tag{A.4}$$

Taking the derivative of (A.4) with respect to M_{T+1} yields

$$\frac{\partial W_T}{\partial M_{T+1}} = (1-M_T)\frac{(1-\mu)^2}{\mu k}u + \beta V'_{0,T+1} - \beta\left(V'_{0,T+1} - V'_{1,T+1}\right) - M_{T+1}\beta\left(V'_{0,T+1} - V'_{1,T+1}\right). \tag{A.5}$$

It can be verified that

$$V_{1,T+1} - V_{0,T+1} = \frac{(1-\mu)}{1-\beta+(1-\mu)\frac{\beta}{k}}\left(\frac{1-M_{T+1}}{k}u + \frac{M_{T+1}}{k}c\right).$$

With $m_0 = (1-M_{T+1})/k$ and $m_1 = M_T/k$, this can be rewritten

$$V_{1,T+1} - V_{0,T+1} = \frac{(1-\mu)}{1-\beta+(1-\mu)(m_0+m_1)\beta}\left(m_{0,T+1}u + m_{1,T+1}c\right)$$

since $m_0 + m_1 = 1/k$.

$V_{0,T+1}$ can be obtained from writing the steady-state equations for V_0 and V_1 ((A.2) and (A.3)) as

$$V_0\left(1-\beta+\beta(1-\mu)m_1\right) = \frac{\mu}{k}(u-c) + (1-\mu)m_1\left(\beta V_1 - c\right),$$
$$V_1\left(1-\beta+\beta(1-\mu)m_0\right) = \frac{\mu}{k}(u-c) + (1-\mu)m_0\left(\beta V_0 + u\right).$$

Some algebra yields

$$V_0 = \frac{\mu}{k}\frac{(u-c)}{(1-\beta)} + \frac{\beta(1-\mu)^2 m_0 m_1}{(1-\beta)\left(1-\beta+\beta(1-\mu)(m_1+m_0)\right)}(u-c) - \frac{(1-\beta)(1-\mu)m_1}{(1-\beta)\left(1-\beta+\beta(1-\mu)(m_1+m_0)\right)}c.$$

Taking the derivative with respect to M_{T+1} yields

$$V'_{0,T+1} = \frac{(1-2M_{T+1})\beta\frac{(1-\mu)^2}{k^2}(u-c) - \frac{(1-\beta)(1-\mu)}{k}c}{(1-\beta)\left(1-\beta+\beta(1-\mu)(m_1+m_0)\right)}.$$

From (A.5),

$$\left.\frac{\partial W_T}{\partial M_{T+1}}\right|_{M_T = M_{T+1} = \hat{M}} = 0$$

implies

$$\left(1-\hat{M}\right)\frac{(1-\mu)^2}{\mu k}u + \frac{\beta^2\frac{(1-\mu)^2}{k^2}(u-c)\left(1-2\hat{M}\right)}{(1-\beta)\left(1-\beta+\beta\frac{1-\mu}{k}\right)} - \frac{(1-\beta)\beta\frac{(1-\mu)}{k}c}{(1-\beta)\left(1-\beta+\beta\frac{1-\mu}{k}\right)}$$
$$+\beta\frac{(1-\mu)}{k}\frac{u+\hat{M}(c-u)}{\left(1-\beta+\beta\frac{1-\mu}{k}\right)} + \beta\frac{(1-\mu)}{k}\frac{\hat{M}(c-u)}{\left(1-\beta+\beta\frac{1-\mu}{k}\right)} = 0.$$

Solving for $\hat{M}$ yields

$$1 > \hat{M} = \frac{(u-c) + u\frac{1-\beta}{\beta}\frac{1-\mu}{\mu}}{2(u-c) + u\frac{1-\beta}{\beta}\frac{1-\mu}{\mu}} \geq \frac{1}{2}.$$