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Impulse Response Identification in DSGE Models

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Abstract

Dynamic stochastic general equilibrium (DSGE) models have become a widely used tool for policymakers. This paper modifies the global identification theory used for structural vectorautoregressions, and applies it to DSGE models. We use this theory to check whether a DSGE model structure allows for unique estimates of structural shocks and their dynamic effects. The potential cost of a lack of identification for policy oriented models along that specific dimension is huge, as the same model can generate a number of contrasting yet theoretically and empirically justifiable recommendations. The problem and methodology are illustrated using a simple New Keynesian business cycle model.

Key words and phrases: identification of DSGE models, impulse response identification, minimal system realisation

JEL: C30, C52

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Knowledge is useful if it helps to make the best decisions.
Jakob Marschak (1953)

1 Introduction

This paper contributes to the global aspect of dynamic stochastic general equilibrium (DSGE) model identification. The focus here is on the global identification of structural shocks that drive the model dynamics. The question we ask is: given the model structure, can we retrieve unique estimates of structural shocks and system responses? It is an important question, because the shock estimates provide interpretation of observed economic variables. Ultimately, it is also the shock estimates that explain model forecasts, and consequently policy recommendations.

The question is motivated by Fukač, Pagan, and Pavlov (2004), but its roots can be found in Marschak's (1953) discussion of usefulness of economic measurements for policy and predictions. Marschak demonstrates that policy makers do not necessarily need to know the complete deep structure of the economy in order to make the best policy decisions. Even limited knowledge of the economic structure might be sufficient to make effective and welfare improving policy decisions. In that spirit, this paper puts aside the question of deep structural identification (e.g., identification of household risk aversion or labour supply elasticity), and concentrates on the identification of structures that guarantee (in a probabilistic sense) a unique explanation of observed data volatility.

Central banks invest a lot of resources into the development of DSGE models. One of the main reasons for this is to be able to conduct coherent structural analysis and forecasting. In addition, having a structural view on economic developments is perceived to be key for credible communication of policy actions to the public. In many ways, DSGE models are new to central bank environment. It is the responsibility of model developers to guarantee the reliability of the information DSGE models provide.

In summary, this paper shows how an existing methodology for structural vector autoregressions (SVAR) developed by Rubio-Ramirez, Waggoner, and Zha (2008) (henceforth RWZ) can be adapted for the identification of invertible DSGE models. The paper deals with solved log-linear DSGE models in a state-space form. The methodology proposed here consists of three steps. The first step is to invert the state-space model into a structural VAR model, which is only possible when the number of observable variables is equal to the number of shocks. We will call this inverted state-space model the *semi-structural model*. The second step is the application of RWZ's SVAR identification theory, which provides a necessary condition for global identification. The third step is to check whether the state-space model is of a minimal realisation. While real economic systems may not have this property, it is desirable for policy-oriented models because it guarantees unique system initial and terminal conditions. If a system is minimal it means that (i) all model variables can be uniquely recovered from observed data, and (ii) unique structural shocks can be recovered from the model variables.

The remainder of the paper is structured as follows. Section two introduces the problem of impulse response identification. Section three looks at the application of the SVAR identification approach to DSGE models. Section four illustrates the results with an example, and section five concludes.

2 The identification problem

A typical DSGE model is non-linear and has forward-looking expectations:

$$0 = \Theta(E_t x_{t+1}, x_t, x_{t-1}, u_t), \tag{1}$$

where $x_t \in \mathbb{R}^{r\times 1}$ is a vector of model variables; $u_t \in \mathbb{R}^{k\times 1}$ is a vector of structural shocks, and $k \leq r$. The shocks are uncorrelated, iid $N(0, \sigma_{u_i}^2)$ for $i = 1, ..., k$. Θ is a non-linear vector function relating the endogenous and exogenous variables by a set of *deep-structural parameters* θ. θ captures microeconomic characteristics of the economic agents in the model (such as their time preferences, risk aversion, frequency of price adjustment, retained earnings, tax rates or inflation target).

We work with the (log)linearized form of model (1), and call this the *structural model*:

$$B_0 x_t = B_1 E_t x_{t+1} + B_2 x_{t-1} + F u_t. \tag{2}$$

B_0, B_1, $B_2 \in \mathbb{R}^{r \times r}$ $(r \times r)$ and $F \in \mathbb{R}^{r \times k}$ are full column rank matrices. The elements of these matrices are functions of the deep-structural parameters θ.

For simplicity at this stage, all model variables are assumed to be measurable. Solving the structural model for the rational expectations (e.g., by the method of undetermined coefficients), we obtain what will be called here the *semi-structural model*:[1]

$$G_0 x_t = G_1 x_{t-1} + u_t$$

$G_0 \in \mathbb{R}^{r \times r}$ defines the contemporaneous relations among the endogenous variables, and $G_1 \in \mathbb{R}^{r \times r}$ captures their dynamics. The elements of the Gs, which are denoted as η, are functions of the deep-structural parameters θ: $\eta = \eta(\theta)$. Note that the semi-structural model is in fact a structural vector autoregressive model of a finite order.

The central question of this paper is: *are there exclusion restrictions on G_0 such that impulse responses are identified?* An impulse response is said to be identified if η can be uniquely estimated from the data.[2]

First, we define the conditions for global and local identification:

Definition 1. *The impulse responses of system (2) are* globally identified *if the set of G_0 and G_1 elements $\eta \in \mathbb{R}$ is not observationally equivalent to another set $\tilde{\eta} \in \mathbb{R}$. The two sets are observationally equivalent if $L(\eta) = L(\tilde{\eta})$, where $L(.)$ is a well behaved loss function.*

Definition 2. *The impulse responses of (2) are* locally identified *if there exists some neighbourhood B in which the set of G_0 and G_1 elements $\eta \in \mathbb{R}$ is not observationally equivalent to another set $\tilde{\eta} \in \mathbb{R} \cap B$.*

[1] We use the term "semi" in order to distinguish the model from an SVAR model.

[2] In this paper we only consider exclusionary restrictions. Other tractable restrictions (such as equality and linear restrictions) are left for further work.

The goal of estimating DSGE models is to pin down the values of the deep structural parameters θ. Why, then, does it make sense to look at the identifiability of η rather than θ? The answer is that it might be the case that η can be uniquely estimated despite θ being unidentified. Uniqueness of the θ parameters is key for policy experiments and welfare analysis, but identification of the θ parameters is sufficient for economic forecasting.

Fukač, Pagan, and Pavlov (2007) discuss that question. The Fisher information matrix for the set of deep structural parameters θ, is given by the variance of the scores: $E\left[\frac{\partial L(\theta)}{\partial \theta}\right]^2$, where $L(.)$ is the likelihood function. By the chain rule the likelihood gradient can be decomposed as

$$\frac{\partial L(\theta)}{\partial \theta} = \frac{\partial L(\theta)}{\partial \eta}\frac{\partial \eta}{\partial \theta}.$$

Thus the information for θ will be the Fisher information for η times the square of $\frac{\partial \eta}{\partial \theta}$. If the latter is singular then the information matrix for θ is also singular, which indicates that some of the parameters in θ are not identified (see, e.g. Iskrev 2009). But note that the singularity may not appear for the information about η.

The identification problem for the semi-structural model is in principle the same as for a structural VAR model. The major difference is that the DSGE model often contains latent variables. As a result, the problem of invertibility arises for the DSGE model. The invertibility property depends on the number of model variables (how many of them we can statistically measure), and the number of exogenous shocks. In the next section we will see that the dimension of shocks is key for invertibility.

3 Identification methodology

Under certain circumstances DSGE and SVAR models are two sides of the same coin. In this section we discuss these circumstances, and show how an existing theory for SVAR models may be applied to (log)linear DSGE models. The section is structured according to the steps involved in the methodology. First, the log-linear model is inverted into an SVAR model. Next the identification of impulse responses is checked, and consequently the identifiability of initial conditions is checked.

When identifying the DSGE impulse responses, we distinguish between the identification of impulse response dynamics, and the identification of shocks and initial conditions. System dynamics after an impulse (speed and profile of convergence) are determined by the size of the parameters in (4). But dynamics of *observed data* are given by the system initial conditions x_0 and the sequence of shocks u_t (their size and qualitative nature). The first of these issues is dealt with by the methodology of RWZ, while the second sits within the concept of minimal system realisation. But both are jointly important for forecasting models, as a model forecast is an impulse response initiated from a proper initial condition.

3.1 Inverting a DSGE model

The model (2) has the minimum state variable (MSV) solution of the form

$$G_0 x_t = G_1 x_{t-1} + Q u_t, \tag{3}$$

where $G_0, G_1 \in \mathbb{R}^{r\times r}$, and $Q \in \mathbb{R}^{r\times k}$ is a full column rank matrix. $G_0 = B_0 - B_1 G_0^{-1} G_1$, $G_1 = B_2$, and $Q = F$.

We can put (3) into state-space form, and estimate it with the Kalman filter. The MSV solution establishes the transition equation:

$$x_t = \mathbf{A} x_{t-1} + \mathbf{B} u_t. \tag{4}$$

$\mathbf{A} = G_0^{-1} G_1$, and $\mathbf{B} = G_0^{-1} F$. The map of the state (model endogenous) variables to their observable counterparts establishes the measurement equation

$$y_t = C x_t. \tag{5}$$

$y_t \in \mathbb{R}^{n\times 1}$ is the vector of observable variables. $C \in \mathbb{R}^{n\times r}$, and $r \geq n$. For simplicity, no measurement errors are assumed in (5). However, the results hold under measurement errors as well. Please note that that MSV form that constitutes the state equation does not guarantee the minimum realisation of a state-space system. We will return to this topic.

The application of RWZ's methodology requires model (4)-(5) to be written in terms of observable variables y_t and their own past values. The state-space model has to be inverted. We will call the result the *semi-structural model*.[3]

In general, there are more state variables in DSGE models than we can actually observe[4]. The first step in deriving the semi-structural forms is to substitute (4) into (5), which gives us:

$$y_t = \mathbf{C}x_{t-1} + \mathbf{D}u_t, \tag{6}$$

where $\mathbf{C} = C\mathbf{A}$ and $\mathbf{D} = C\mathbf{B}$. $\mathbf{D}$ might not be invertible (or left invertible), because it does not necessary have a full column rank. Thus we impose the following assumption.

Assumption 1. *$\boldsymbol{D}$ is invertible, or at least left invertible, i.e. $\boldsymbol{D}^+\boldsymbol{D} = I$.*

This assumption restricts us to state-space models that have the same number of shocks (structural and/or measurement errors) as observable variables, $n = k$.

Assumption 1 is used to solve (6) for $u_t = \mathbf{D}^+ y_t - \mathbf{D}^+\mathbf{C}x_{t-1}$. Plugging that into (4) and re-arranging gives us $x_t = [\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{D}^+\mathbf{C})L]^{-1}\mathbf{B}\mathbf{D}^+ y_t$. By substituting

[3] In the engineering literature, where such inversion comes from, it is called the impulse response function (see Ljung, 1999, Section 4.3). Villaverde *et al.* (2007) study the properties of such a transformation for economic problems.

[4] This creates only a minor complication for the invertibility technique itself. If C is invertible ($k = n$), then it is straightforward to solve for the semi-reduced form. From the state equation (4), x_t is solved and substituted into the measurement equation (5).

$$\begin{aligned}
(I - \mathbf{A}L)x_t &= \mathbf{B}u_t \\
(I - \mathbf{A}L)C^{-1}y_t &= \mathbf{B}u_t \\
\mathbf{B}^+C^{-1}y_t &= \mathbf{B}^+\mathbf{A}C^{-1}y_{t-1} + u_t \\
A_0 y_t &= A_1 y_{t-1} + u_t
\end{aligned}$$

with $A_0 = \mathbf{B}^+C^{-1}$ and $A_1 = \mathbf{B}^+\mathbf{A}C^{-1}$. If $\mathbf{A}$ is a stable matrix – which is almost always the case as it comes from the rational expectations solution – the state-space model can be represented as a structural VAR(1).

this back into the measurement equation (6) we obtain an infinite order SVAR representation of (4)-(5)

$$\mathbf{D}^{+}y_t = \mathbf{D}^{+}\mathbf{C}\sum_{j=0}^{\infty}(\mathbf{A}-\mathbf{B}\mathbf{D}^{+}\mathbf{C})^{j}\mathbf{B}\mathbf{D}^{+}y_{t-j-1}+u_t. \tag{7}$$

In summary, if the dimensions of x_t, y_t and u_t are the same than the semi-structural form is in fact a finite order SVAR model. If the dimension of x_t is higher than those of y_t and u_t, the semi-structural model corresponds to the infinite order SVAR (7).

3.2 Impulse response identification

The heart of the impulse response identification lies in the theory of RWZ. In this section we summarise the key features of their theory (for details see RWZ, 2008, Section II), and extend it to the case where SVAR representations have more shocks than observable variables.

RWZ's methodology requires (7) to be transposed such that individual equations are in columns.

$$y_t' A_0 = \mathbf{y}_t' A_{+}(L) + u_t', \tag{8}$$

where $\mathbf{y}_t = [y_{t-1}y_{t-2}...y_{-\infty}]'$. $\mathbf{A}_{+}(L)' = [\mathbf{I}-(\mathbf{A}-\mathbf{B}\mathbf{D}^{+}\mathbf{C})L]^{-1}\mathbf{B}\mathbf{D}^{+}$ is an infinite polynomial. $A_0 = (\mathbf{D}^{+})' = [(CG_0^{-1}F)^{-1}]'$ is an $n \times n$ matrix capturing the contemporaneous relationships among observed endogenous variables implied by the theoretical model. Villaverde *et al.* (2007) show that if $(\mathbf{A}-\mathbf{B}\mathbf{D}^{+}\mathbf{C})$ is stable then y_t is a bounded sequence.[5]

We depart from RWZ by ignoring assumption 1 for a moment and assume instead that A_0 is of dimension $k \times n$. This is not an invertible matrix, but it has full row rank so it is right invertible. A reduced form representation for this matrix can be obtained by taking the Moore-Penrose pseudoinverse.[6] If there are n observable

[5] See footnote 2 of this paper. If $n = k = r$ and C is an identity matrix then (8) shrinks to SVAR(1), which is the MSV solution (3).

[6] The key computational rules with the pseudoinverse operator are summarised in Appendix E.

variables in the model then the full row rank assumption holds. The reduced form is then

$$y_t' = \mathbf{y}_t' B + u_t',$$

where $B = A_+ A_0^+$ is of dimension $m \times n$, and $u_t' = \varepsilon_t' A_0^+$ is a $1 \times n$ vector of reduced structural shocks. Note that the dimension of shocks corresponds to the number of y_t. We have as many reduced form shocks as observable variables. $E[u_t u_t'] = E[A_0^{+\prime} \varepsilon_t \varepsilon_t' A_0^+] = (A_0 A_0')^+ = \Sigma$, where Σ is an $n \times n$ variance-covariance matrix of reduced form shocks.

The first key theorem in RWZ is about the observability equivalence.

Theorem 1 (Observability equivalence)**.** *Two sets of structural parameters in (7), (A_0, A_+) and $(\tilde{A}_0, \tilde{A_+})$, are observationally equivalent if and only if there exists a $k \times k$ orthogonal matrix P such that $A_0 = \tilde{A}_0 P$ and $A_+ = \tilde{A}_+ P$.*

Proof. See Appendix A. □

In order to check identifiability of the structure (A_0, A_+) we need to be able to represent the parameter restrictions. Here we stick with the exclusionary restrictions as studied by RWZ. For $1 \leq j \leq k$ and $f(A_0, A_0) = [A_0 \quad A_+]'$ of dimension $g \times k$, where $g = n + m$, RWZ define matrix

$$M_j(f(A_0, A_+)) = \left[\begin{array}{cc} \multicolumn{2}{c}{Q_j f(A_0, A_+)} \\ \hline I_j & 0 \end{array} \right]$$

where I_j is a $j \times j$ identity matrix, and 0 is a $j \times k - j$ zero matrix. The linear restrictions can be represented by $g \times g$ matrices Q_j for $1 \leq j \leq k$. Each matrix Q_j has rank q_j. The structural parameters (A_0, A_+) satisfy the restrictions if and only if

$$Q_j f(A_0, A_+) e_j = 0,$$

where e_j is the j^{th} column of the $k \times k$ identity matrix I_k. The ordering of Q_j is such that

$$q_1 \geq q_2 \geq \ldots \geq q_k.$$

The ordering of restrictions is important for the identification check as it utilises the recursive nature of the model. Shocks estimates are used as extra instrumental variables for identification. As discussed in RWZ, this is the major difference from identifying a system of linear equations in classical econometrics. In SVAR models residuals are allowed to be correlated whereas in the classical linear system residuals are orthogonal.

Theorem 2 (The general rank condition)**.** *If* $(A_0, A_+) \in R$ *and* $M_j(f(A_0, A_+))$ *is of rank k for all* $1 \leq j \leq k$*, then the SVAR is globally identified at* (A_0, A_+).

Proof. This theorem is adjusted Theorem 5 from RWZ. See Appendix B. □

Finally, having defined the SVAR representation of the DSGE model (7), we can apply theorem 2. Since A_+ is an infinite order polynomial, $f(A_0, A_+)$ is also of infinite size. However, to apply the theorem we can focus on a finite order model with $j = 1$. $A_2' = \mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})\mathbf{BD}^+$, and because the matrices $A_j' = \mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})^j\mathbf{BD}^+$ for $j > 1$ are combinations of A_2, the rank of A_+ will be equal to the rank of A_2. Therefore it is sufficient to construct $f(A_0, A_+)$ as

$$f(A_0, A_1, A_2) = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} [(\mathbf{D}^+]' \\ [\mathbf{D}^+\mathbf{CBD}^+]' \\ [\mathbf{D}^+\mathbf{C}(\mathbf{A} - \mathbf{BD}^+\mathbf{C})\mathbf{BD}^+]' \end{bmatrix}.$$

Given $f(A_0, A_1, A_2)$ we can form Q_j to represent zero restrictions, and correspondingly $M_j(f(A_0, A_+))$ for all $1 \leq j \leq k$.

Overall, the strength of theorem 2 is that it applies globally. The theorem gives a necessary condition, but if the number of exclusionary restrictions is equal to $(n-1)/2$, it also provides a sufficient condition.

3.3 Checking minimal system realisation

The condition of minimal realisation may appear restrictive, as one may believe that we live in an uncontrollable world, but from the perspective of a decisionmaker, it is appealing to work with the model structures that satisfy such a condition.[7] We need a solid information ground to make best decisions. Unique initial conditions, unobservable variables and structural shocks estimates provide such a ground. Then there is only one degree of freedom to tell an economic story based on the model. Its uniqueness guarantees that the model's interpretation of the past economic development will not dramatically change, and stays consistent and credible over time. In the economics we often work with unobservable concepts like real marginal costs, risk premiums, cost-push shocks or monetary policy shocks. In order to use those concepts to interpret the dynamics of variables like inflation or interest rate we need to know their reliable estimates.

Definition 3 (Observability)**.** *The state-space system $\{A,B,C,D\}$ is called observable if the observability matrix $O_n(C,A)$ has rank n,*

$$O_T(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}.$$

If the system is observable, then we can always solve for the initial state x_0 from a given set of shocks u_t (typically assumed to be zero) and observables y_t, for $t \geq 0$.

Definition 4 (Controllability)**.** *The state-space system $\{A,B,C,D\}$ is called controllable if the controllability matrix $C_n(B,A)$ has rank n,*

$$C_T(B,A) = \begin{bmatrix} B & AB & \dots & A^{T-1}B \end{bmatrix}.$$

[7] In contrast to the electrical engineering literature where the control theory originated, economics introduces concepts for which economists do not have measurable counterparts. Even though it is a well known feature of dynamic systems, it seems that the minimal system property is often omitted in many economic applications. One can often see DGSE models with twice as many model endogenous variables than observed time series on which to estimate the model. Unique estimation of unobservable endogenous variables is part of the identification problem we are interested in. Having properly identified initial conditions for all endogenous variables (both observable and unobservable) is necessary for unique forecasts.

If the system is controllable then for any initial state it is possible to design a unique set of shocks that will lead to a desired trajectory of states x_t.

Theorem 3 (System minimal realisation)**.** *The system* $\{A,B,C,D\}$ *is minimal if it is observable and controllable.*

Proof. See Kalman (1962) for the proof. □

In engineering literature the problem of minimal realisation is described as: given some data about linear time invariant system, find a state space description of minimal size that describes the given data (e.g. De Schutter 2000, p.332). In the economics, we have The following theorem states how the minimal realisation problem is related to the initial condition identification.

Theorem 4. *If the order of the state-space system is minimal then we can uniquely recover the structural shocks* $\{u_t\}_{t=1}^T$ *and state variables* $\{x_t\}_{t=0}^T$*.*

Proof. The problem can be broken up into two parts. First, if we know $\{y_t\}_{t=1}^T$ can we get a unique x_0, that is a unique $\{x_t\}_{t=0}^T$ that leads to x_0? This is equivalent to checking the observability condition. Second, knowing x_0 and $\{x_t\}_{t=1}^T$, can we get a unique sequence of exogenous shocks $\{u_t\}_{t=1}^T$ that explains such a trajectory? This is equivalent to checking the controllability condition.

Looking at the first problem, we solve the following system of equations:

$$\begin{aligned}
y_1 &= Cx_0 + Du_1 \\
y_2 &= CAx_0 + CBu_1 + Du_2 \\
y_3 &= CA^2x_0 + CABu_1 + CBu_2 + Du_3 \\
&\vdots \\
y_T &= CA^{T-1}x_0 + CA^{T-2}Bu_1 + \ldots + CBu_{T-1} + Du_T
\end{aligned}$$

In matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} - \begin{bmatrix} D & 0 & 0 & 0 & \dots & 0 \\ CB & D & 0 & 0 & \dots & 0 \\ CAB & CB & D & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ CA^{T-2}B & CA^{T-3}B & CA^{T-4}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_T \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix} x_0 \tag{9}$$

If the matrix on the right-hand side of equation (9) (the observability matrix) is left-invertible (i.e. it has full column rank), then the system can be uniquely solved for x_0.

Looking at the second problem, we know x_0 and solve the system for the unique realisation of $\{u_t\}_{t=1}^T$:

$$\begin{aligned} x_1 &= Ax_0 + Bu_1 \\ x_2 &= A^2 x_0 + ABu_1 + Bu_2 \\ x_3 &= A^3 x_0 + A^2 Bu_1 + ABu_2 + Bu_3 \\ &\vdots \\ x_T &= A^T x_0 + A^{T-1}Bu_1 + \ldots + ABu_{T-1} + Bu_T \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_T \end{bmatrix} - \begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^T \end{bmatrix} x_0 = \begin{bmatrix} B & 0 & 0 & \dots & 0 \\ AB & B & 0 & \dots & 0 \\ A^2B & AB & B & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{T-1}B & A^{T-2}B & A^{T-3}B & \dots & B \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_T \end{bmatrix} \tag{10}$$

The solution is unique if the matrix on the right-hand side is invertible. It is invertible if the controllability matrix $C_T(B,A)$ has full column rank. Thus if the state-space model is minimal we get a unique trajectory for both $\{x_t\}_{t=0}^T$ and $\{u_t\}_{t=1}^T$ □

4 An illustration

In this section we will illustrate the use of the identification methodology on a simplified version of the closed economy New Keynesian business cycle model. The New Keynesian model can be summarised by the following three equations:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(\varphi+\nu)(1-\zeta\beta)(1-\zeta)}{\zeta}\xi_t + u_{S,t}, \tag{11}$$

$$\xi_t = E_t \xi_{t+1} + \frac{1}{\varphi}[r_t - E_t(\pi_{t+1})] + u_{D,t}, \tag{12}$$

$$r_t = \phi_r r_{t-1} + (1-\phi_r)(\phi_\pi \pi_t + \phi_\xi \xi_t) + u_{r,t}. \tag{13}$$

The Phillips curve (11) is firms' linearized pricing rule, where π_t is the aggregate price level inflation rate. The IS curve (12) is households' linearized Euler equation capturing the output ξ_t. The nominal side of the economy is controlled by the central bank's interest rate rule (13), where r_t is the nominal interest rate set in period t, and $u_{S,t}$, $u_{D,t}$, and $u_{r,t}$ are the supply (cost-push) shock, demand shock, and monetary-policy shock, respectively. All shocks are iid $N(0,\sigma^2_{u_i})$ for all $i=\{S,D,r\}$. The model's deep structural parameters (earlier denoted as θs) are $0<\beta<1$, $\nu>0$, $\varphi>0$, $\zeta>0$, $0\leq\phi_r<1$, and ϕ_π and ϕ_ξ are such that there exists a unique and stable equilibrium.

We can immediately see that the parameters ν and ζ cannot be identified, as ζ comes from a quadratic equation. Following the discussion in section 2, this is not disturbing because unique values of the deep structural parameters are not very important here. It is their product $\frac{(\varphi+\nu)(1-\zeta\beta)(1-\zeta)}{\zeta}$ that determines the impulse response function that we are interested in.

Solving the model for rational expectations, we end up with the law of motion for ξ_t, π_t, and r_t. We look for the MSV representation such that we can construct A_0 and A_+ for the identification methodology. The MSV representation of (11)-(13) is[8]

$$A_0 y_t = A_1 y_{t-1} + u_t,$$

[8] A derivation is outlined in Appendix C.

where $y_t = \begin{bmatrix} \pi_t & \xi_t & r_t \end{bmatrix}'$, $u_t = \begin{bmatrix} u_{D,t} & u_{S,t} & u_{r,t} \end{bmatrix}'$, and

$$A_0' = \begin{pmatrix} a_{0,11} & 0 & a_{0,31} \\ a_{0,12} & a_{0,22} & a_{0,32} \\ a_{0,13} & a_{0,23} & a_{0,33} \end{pmatrix}, \quad A_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{1,33} \end{pmatrix}.$$

We now need to determine whether the structure of A_0 and A_1 is such that the value of the semi-structural parameters $a_{0,ij}$ and $a_{1,ij}$ for all $i,j = 1,2,3$ can be uniquely pinned down by the data.

4.1 The same number of endogenous variables, observables, and shocks

Let us start with the simplest case where all model variables are assumed to be observed and the number of shocks are equal to the number of observables, i.e. $n = k = r = 3$. This is the simplest case, because **C** is an identity matrix and the MSV solution directly yields an SVAR(1) model. No DSGE model invertibility is required and thus we have a straightforward application of RWZ's theory.

First, we form the transformation $f(A_0', A_1')$ by stacking A_0' and A_1'

$$f(A_0', A_1') = \begin{pmatrix} A_0' \\ A_1' \end{pmatrix}.$$

Note that the matrices A_0 and A_1 are transposed, and individual equations are captured in columns.

Second, we re-order the equations by the descending number of exclusionary restrictions as required by theorem 2. We swap the IS curve with the Phillips curve as the IS curve has four exclusionary restrictions while the Phillips curve has only three. By swapping the second column of $f(.)$ with the first one, we get

$$f(A'_0, A'_1) = \left[\begin{array}{ccc} 0 & a_{0,11} & a_{0,31} \\ a_{0,22} & a_{0,12} & a_{0,32} \\ a_{0,23} & a_{0,13} & a_{0,33} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{1,33} \end{array}\right]$$

Third, we represent the zero restrictions in $f(A'_0, A'_1)$ by Q_j matrices that form the nullspace with $f(A'_0, A'_1)$. Each Q_j captures the exclusionary restrictions in the j's column of $f(A'_0, A'_1)$. For the first column, the IS curve, we have Q_1

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the second and third column, the Phillips curve and the policy rule respectively, we form

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The fourth and final step is to construct the matrices $M_j(f(A'_0, A'_1))$ from theorem 2 for all $j = 1, 2, 3$. Skipping the zero rows, we get

$$M_1 = \left(\begin{array}{ccc} 0 & a_{0,21} & a_{0,31} \\ 0 & 0 & a_{1,33} \\ \hline 1 & 0 & 0 \end{array}\right), M_2 = \left(\begin{array}{ccc} 0 & 0 & a_{1,33} \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, each M_j represents individual model equation that are ordered as the columns in $f(.)$. The rank of M_j can be interpreted in a partial way. If $rank(M_j) = n$, one concludes that the shock associated with the j^{th} equation is identified. Clearly, $rank(M_j) = 3$ for all j here, and thus we can conclude that the semi-structural model *is identified*. Note that the identification comes from the lagged interest rate r_{t-1} in the policy rule. If $\phi_r = 0$ then the model does not produce enough instruments to identify the Phillips curve and the rank condition would be violated, $rank(M_2) = 2$.

4.2 More endogeneous variables than observables and shocks

Now let us look at the case where the number of observable variables is less than the number of model variables. This will require the DSGE model to be inverted, and we will show that the location of structural shocks matters for identification in such cases.

Let us reduce the number of shocks and observable variables.[9] The shocks and observables are carefully chosen in the examples below in order to serve the illustration purposes best.

4.2.1 No monetary policy shock: $\sigma^2_{u_R} = 0$

The output ξ (or output gap) is stochastic but unobservable in this economy. We only observe the interest rate and inflation. We also assume that the model developer, in order to keep the number of shocks equal to the number of observables, assumes that there is no policy error in setting the interest rate according to (13), $\sigma^2_{u_R} = 0$. Thus the observed volatility of the interest rate will be due to supply and demand shocks.

First, we have to invert the DSGE model into an SVAR. Because now $n = k < r$,

[9] This exercise is equivalent to compounding the shocks, so that their number is reduced to n in the measurement equation (6).

the SVAR will be of infinite order. Setting $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and substituting them into (8), we obtain A_0 and A_+ such that

$$f(A_0', A_1') = \begin{pmatrix} + & + \\ + & + \\ \hline 0 & 0 \\ + & + \end{pmatrix}.$$

For $j > 1$ $A_j = 0$. For interested readers, the detailed derivation is in Appendix C.

Having $f(A_0', A_1')$ one proceeds in exactly the same way as before, by constructing Q_j and then $M_j(f(A_0, A_1))$ for $j = 1, 2$, which gives

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We find that $rank(M_1) = 1$ and thus the supply shock $u_{S,t}$ associated with the Phillips curve is not identified.

4.2.2 No demand shock: $\sigma_{u_D}^2 = 0$

In this example we still do not observe the output ξ_t but the IS curve is now a deterministic relation, $\sigma_{u_D}^2 = 0$. Instead the interest rate rule will be subject to stochastic errors. Because we have $n = k = 2$ and $r = 3$, the inverted DSGE model will again be of infinite order.

Following the four-step procedure above, the $f(A_0', A_1')$ representation is

$$f(A_0', A_1') = \begin{pmatrix} + & + \\ + & + \\ \hline 0 & 0 \\ 0 & + \end{pmatrix}.$$

Correspondingly, $M_j(f(A_0,A_1))$ for $j = 1,2$ turns out to be

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & + \\ \hline 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The location of shocks matters. Because $rank(M_1) = rank(M_1) = 2$ both supply and policy shocks can be identified from the data and thus the impulse response of observable variables to structural shocks can also be uniquely informed by the data. Changing the assumption about the shocks results in their identifiability.

4.3 Implications for forecasting models

This paper has been motivated by the need for applied models to be identified. Therefore it is natural to ask what it means practically when a model does not have identified impulse responses? In table 1 we report two different sets of parameter estimates of (11)-(13). ϕ_r is restricted to zero and thus the model is not identified. Estimating such a model on US of inflation and interest rate data gives the two sets of maximum likelihood estimates, both consistent with the same value of the likelihood function of 63.31. Abstracting from the economic interpretation of the estimates *per se*, there is no way to distinguish which set of estimates is preferred by the data. Thus, the model provides two different recommendations for setting the policy instrument as can be seen in figures 1 and 2.

Figure 1 shows the historical shock decompositions of the US inflation and interest rate under parameterization 1 from table 1. The dashed and dotted lines plot the contribution of supply and policy shocks respectively, to the development of the inflation rate (in the top panel), and interest rate (bottom panel). Summing up the shock contributions gives the value of the observed series. In the right panels of figure 1 we show the forecasts from 2001:Q4 onwards where inflation gradually returns to its steady state value. As a result of such sluggish price adjustment, the model recommends only a gradual increase in the policy rate. Starting from about

70 bp below its neutral levels the model recommends approximately three 25bp hikes for the interest rate to return the economy to the steady state.

Table 1: Alternative parametrizations implying the same likelihood

	Parametrization 1	Parametrization 2
β	0.99	0.99
ϕ	0.87	3.72
κ	0.65	6.14
θ_π	2.68	1.85
θ_ξ	1.05	0
σ_{uS}	0.46	0.76
σ_{ur}	0.95	0.56
Likelihood	63.31	63.31

The parameters values are estimates from the New Keynesian model using US inflation and 3M T-bills data from 1982:q1 to 2001:q4. The different values were obtained by providing different initial conditions for the estimation algorithm, $\kappa = \frac{(\varphi+\nu)(1-\zeta\beta)(1-\zeta)}{\zeta}$.

Before drawing any conclusions from this, we should bear in mind that both parameterizations 1 and 2 are associated with the same data likelihood. Inspecting the same graphs in figure 2, prices seem to be much more flexible under parameterization 2. They are predicted to rise back to their steady state values in about one quarter. Monetary policy must follow and close its expansionary stance very quickly in order to avoid causing inflationary pressures in the future. The model recommends a hike of about 40 bp in one quarter to offset that. This is quite a difference in comparison to the three 25bp increases of interest rates recommended before.

This is a simple illustration of risks associated with unidentified models. In large models with more complex policy transmission mechanisms, we may end up with even more contradictory policy prescriptions – under one parameterization the policy rate might be perceived as too loose and under another too tight. The policy errors might be qualitatively different.[10]

[10] This was the experience for instance in developing KITT, the Reserve Bank of New Zealand forecasting model (see Beneš, *et al.* 2009).

Models are often estimated by Bayesian methods that downplay the problem of structural identification. However, the modeller is still interested in updating priors by data information. If the model structure is such that it prevents data from speaking, one has to rely on the priors. The question is are the priors strong enough, or do we only have them in order to be able to run something appearing like estimation? If the prior is not strong then posterior estimates suffer exactly the same problem illustrated in figures 1 and 2.

Figure 1: Forecast with model Parametrization 1

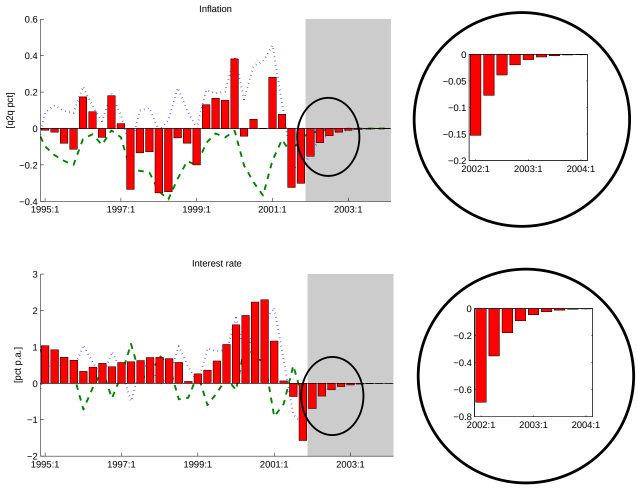
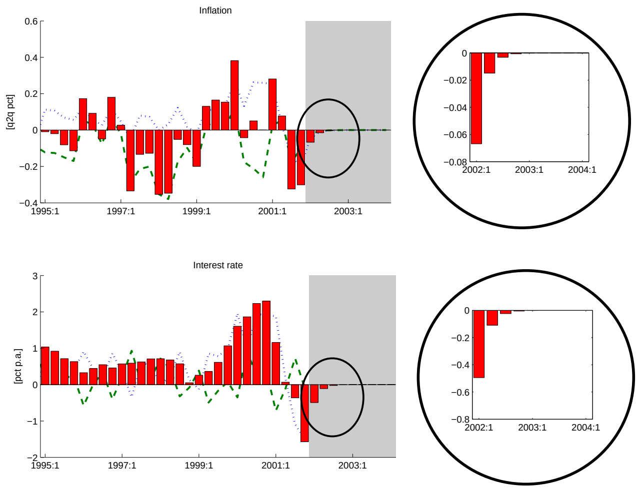


Figure 2: Forecast with model Parametrization 2



5 Conclusion

This paper has shown that the SVAR identification methodology developed by Rubio-Ramirez, Waggoner and Zha (2008) can be applied to DSGE models with unobservable variables. We used the RWZ methodology to determine whether the model's semi-structural form is globally identifiable, with the aim of estimating unique impulse responses. If there is no other observationally equivalent set of structural shocks that would explain the data, the model is said to have unique (identified) impulse responses. The methodology consists of a few matrix operations and evaluations and is straightforward to apply, particularly to large scale models. Because no evaluation of likelihood functions is involved the methodology is computationally cheap. It takes only seconds to evaluate the objectives. It can also provide useful information for DSGE model developers. There are many types of structural shocks that can be used to make the dynamic model stochastic and economic theory does not always provide guidance which to choose. Thus shock identifiability may serve as one criteria for a discriminating among them.

This paper only scratches the surface of the DSGE model identification problem. Rothenberg (1971) sets general conditions for structural model identfiability, but there is still a lack of techniques that allow these conditions to be evaluated in practice. We leave the question of global identification of deep structural parameters and the problem of identifiability of DSGE models containing unit roots for the future research.

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A Proof of Theorem 1

Proof. If $A_0 = \tilde{A}_0 P$ and $A_+ = \tilde{A}_+ P$, then

$$B = A_+ A_0^+ = \tilde{A}_0 P P^{-1} \tilde{A}_0^+ = \tilde{A}_+ \tilde{A}_0^+ = \tilde{B}$$

$$\Sigma = (A_0 A_0')^+ = (\tilde{A}_0 P P' \tilde{A}_0')^+ = (\tilde{A}_0 \tilde{A}_0')^+ = \tilde{\Sigma}.$$

If they are observationally equivalent then $A_+ A_0^+ = \tilde{A}_+ \tilde{A}_0^+$ and $(A_0 A_0')^+ = (\tilde{A}_0 \tilde{A}_0')^+$. From the latter it follows that

$$\begin{aligned}
(A_0 A_0')^+ &= (\tilde{A}_0 \tilde{A}_0')^+ \\
A_0^{+\prime} A_0^+ &= \tilde{A}_0^{+\prime} \tilde{A}_0^+ \\
A_0' A_0^{+\prime} A_0^+ &= A_0' \tilde{A}_0^{+\prime} \tilde{A}_0^+ \\
A_0^+ &= A_0' \tilde{A}_0^{+\prime} \tilde{A}_0^+ \\
A_0' (A_0 A_0')^{-1} &= (\tilde{A}_0^+ A_0)' \tilde{A}_0^+ \\
A_0' &= (\tilde{A}_0^+ A_0)' (\tilde{A}_0^+ A_0) A_0' \\
(A_0 A_0')' [(A_0 A_0')']^{-1} &= (\tilde{A}_0^+ A_0)' (\tilde{A}_0^+ A_0) (A_0 A_0')' [(A_0 A_0')']^{-1} \\
I &= (\tilde{A}_0^+ A_0)' (\tilde{A}_0^+ A_0).
\end{aligned}$$

Therefore $P = \tilde{A}_0^+ A_0$ is orthogonal and $\tilde{A}_0 P = A_0$. That is

$$\begin{aligned}
P &= \tilde{A}_0^+ A_0 \\
\tilde{A}_0 P &= \tilde{A}_0 \tilde{A}_0^+ A_0 \\
\tilde{A}_0 P &= \tilde{A}_0 \tilde{A}_0' (\tilde{A}_0 \tilde{A}_0')^{-1} A_0 \\
\tilde{A}_0 P &= A_0.
\end{aligned}$$

Using this result for $A_+ A_0^+ = \tilde{A}_+ \tilde{A}_0^+$, we obtain

$$\begin{aligned}
A_+ A_0^+ &= \tilde{A}_+ \tilde{A}_0^+ \\
A_+ A_0' (A_0 A_0')^{-1} &= \tilde{A}_+ \tilde{A}_0^+ \\
A_+ A_0' &= \tilde{A}_+ \tilde{A}_0^+ A_0 A_0' \\
A_+ (A_0 A_0')' &= \tilde{A}_+ \tilde{A}_0^+ A_0 (A_0 A_0')' \\
A_+ (A_0 A_0')' [(A_0 A_0')']^{-1} &= \tilde{A}_+ \tilde{A}_0^+ A_0 (A_0 A_0')' [(A_0 A_0')']^{-1} \\
A_+ &= \tilde{A}_+ \tilde{A}_0^+ A_0 \\
A_+ &= \tilde{A}_+ P.
\end{aligned}$$

□

B Proof of Theorem 2

Proof. With a minor modification, the proof is the same as in RWZ (2007, p.15). Let $q_j = Pe_j - p_{jj}e_j$, where $P = \tilde{A}_0^{+}A_0$ is a $k \times k$ orthogonal matrix, p_j is the first column of P with non-zero off-diagonal elements, e_j is the j^{th} column of an identity matrix I_k. To prove the theorem it is sufficient to show that the rank of $M_j(f(A_0,A_0))$ is strictly less than k. Since $q_j \neq 0$, it suffices to show that $M_j(f(A_0,A_+))q_j = 0$. Because both (A_0,A_+) and (A_0P,A_+P) are in R, by construction of Q_j it holds that $Q_j f(A_0,A_+)q_j = 0$. Thus the upper block of $M_j(f(A_0,A_+))$ is zero. The lower block $[I \quad 0]q_j$ is also equal to zero, because I is a $j \times j$ and first j elements of e_j are zero. □

C Solution to examples in Section 4

Case 1: $r = n = k$

The matrix form of model (11)-(13) is

$$x_t' B_0 = x_{t-1} C + (E_t x_{t+1})' D + u_t',$$

where $x_t = [\pi_t \quad \xi_t \quad r_t]$, B_0, C, and D are 3×3 matrices of the semi-structural form parameters, and u_t is iid $N(0,1)$.

The MSV solution to the model is

$$x_t' A_0 = x_{t-1} A_1 + u_t',$$

where

$$\begin{aligned} B_0 - C A_0^{-1} D &= A_0, \\ A_1 &= C. \end{aligned}$$

Now, the task is to characterize the structure of A_0. We know that

$$B_0 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix}, D = \begin{pmatrix} + & + & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}.$$

Let $A_0^{-1} = \begin{pmatrix} a_{0,11}^* & a_{0,12}^* & a_{0,13}^* \\ a_{0,21}^* & a_{0,22}^* & a_{0,23}^* \\ a_{0,31}^* & a_{0,32}^* & a_{0,33}^* \end{pmatrix}$ and substitute into the MSV solution to get an idea of what the structure of A_0 looks like. We can then apply the counting rule from RWZ's paper. We get

$$\begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ + & + & + \end{pmatrix} \begin{pmatrix} + & + & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{pmatrix} = A_0,$$

and from there $A_0 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ + & + & + \end{pmatrix}$.

Now we can apply theorem 2 to check the general rank condition.

$$f(A_0, A_1) = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} + & 0 & + \\ + & + & + \\ + & + & + \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix} = \begin{pmatrix} 0 & + & + \\ + & + & + \\ + & + & + \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we have to check the rank condition:

$$M_1 \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} & Q_1 \left(\begin{matrix} A_0 \\ A_1 \end{matrix} \right) & \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & + & + \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_2 \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} & Q_1 \left(\begin{matrix} A_0 \\ A_1 \end{matrix} \right) & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We can see that $rank(M_1) = 3$ and $rank(M_2) = 3$, and thus we can conclude that the model is globally identified.

Case 2: $n = k < r$

A.) No monetary policy shock: $\sigma^2_{u_R} = 0$

In this exercise we assume that the output gap ξ is unobservable, and that there is no monetary policy shock $u_{R,t}$. The structural model when solved then takes the form

$$A_0 x_t = A_1 x_{t-1} + F u_t.$$

$$A_0 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ + & + & + \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} y_t &= C x_t \\ &= C A_0^{-1} A_1 x_{t-1} + C A_0^{-1} F u_t \\ &= \mathbf{C} x_{t-1} + \mathbf{D} u_t \end{aligned}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 & + \\ 0 & 0 & + \end{pmatrix}, \mathbf{D} = \begin{pmatrix} + & + \\ + & + \end{pmatrix}.$$

Solving for an SVAR representation of the DSGE model we get

$$A_0 y_t = A_1 y_{t-1} + u_t$$

$$A_0 = \mathbf{D}^+ = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, A_1 = \mathbf{D}^+\mathbf{CBD}^+ = \begin{pmatrix} 0 & + \\ 0 & + \end{pmatrix}.$$

$$f(A_0', A_1') = \begin{pmatrix} + & + \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} Q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$M_1 = \left(\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \end{array}\right), M_2 = \left(\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right).$$

We can see that $rank(M_1) = 1$ and $rank(M_2) = 2$. Thus, the model is *not* identified.

B.) No demand shock: $\sigma^2_{u_D} = 0$

We assume the same setting as in Case 2 A, with the only difference being that there is no demand shock in the IS curve $u_{D,t}$, but there is a monetary policy shock, $u_{R,t}$.

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 & + \\ 0 & 0 & + \end{pmatrix}, \mathbf{B} = \begin{pmatrix} + & + \\ + & + \\ + & + \end{pmatrix}.$$

Similarly to above, $A_0 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 0 & + \end{pmatrix}$.

$$f(A_0', A_1') = \begin{pmatrix} + & + \\ + & + \\ 0 & 0 \\ 0 & + \end{pmatrix}, Q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $M_1 = \left(\begin{array}{cc} 0 & 0 \\ 0 & + \\ \hline 1 & 0 \end{array}\right)$, and $M_2 = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right)$. We can immediately see that $rank(M_1) = 2$ and $rank(M_2) = 2$. The model *is* identified.

D Matrix pseudoinverse

Definition 5 (Matrix pseudoinverse)**.** *For a matrix A whose elements are real numbers, its pseudoinverse A^+ is a unique transformation, which meets the following criteria:*

$$
\begin{aligned}
AA^+A &= A; \\
A^+AA^+ &= A^+; \\
(AA^+)' &= AA^+; \\
(A^+A)' &= A^+A;
\end{aligned}
$$

Some useful properties are:

- Pseudoinversion is reversible: $(A^+)^+ = A$;
- $(A')^+ = (A^+)'$;
- $A^+ = A^+A^{+\prime}A'$;
- $A^+ = A'A^{+\prime}A^+$;
- If A is of full column rank then $A^+ = (A'A)^+A'$, and $A^+A = I$; A^+ is left inverse of A;
- If A is of full row rank then $A^+ = A'(AA')^+$, and $AA^+ = I$; A^+ is right inverse of A;
- If A is square, non-singular matrix then $A^+ = A^{-1}$.