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The state space representation and estimation of a time-varying parameter VAR with stochastic volatility

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Abstract

To capture the evolving relationship between multiple economic variables, time variation in either coefficients or volatility is often incorporated into vector autoregressions (VARs). The state space representation that links the transition of possibly unobserved state variables with observed variables is a useful tool to estimate VARs with time-varying coefficients or stochastic volatility. In this paper, we discuss how to estimate VARs with time-varying coefficients or stochastic volatility using the state space representation. We focus on Bayesian estimation methods which have become popular in the literature. As an illustration of the estimation methodology, we estimate a time-varying parameter VAR with stochastic volatility with the three U.S. macroeconomic variables including inflation, unemployment, and the long-term interest rate. Our empirical analysis suggests that the recession of 2007-2009 was driven by a particularly bad shock to the unemployment rate which increased its trend and volatility substantially. In contrast, the impacts of the recession on the trend and volatility of nominal variables such as the core PCE inflation rate and the ten-year Treasury bond yield are less noticeable.

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1 Introduction

Vector autoregressions (VARs) are widely used in macroeconomics to detect comovements among multiple economic time series. In a nutshell, VARs regress each time series onto various lags of multiple time series included in the model. When coefficients are assumed to be stable, each equation in a VAR becomes an example of a multiple linear regression. In the simplest form, error terms in the VAR are assumed to have constant variances.

While convenient, assuming time-invariant coefficients and variances turns out to be quite restrictive in capturing the evolution of economic time series. For example, U.S. business cycle dynamics and monetary policy have changed substantially over the post-war period. To describe these changes in the VAR framework requires one to allow shifts in coefficients or volatility (e.g. Canova and Gambetti (2009), Clark (2009), Cogley and Sargent (2005), Cogley, Primiceri, and Sargent (2010), Primiceri (2005), Sims and Zha (2006)).

When time variation is introduced to either coefficients or volatility in a VAR, the state space representation of the VAR is typically used in empirical analysis to estimate unobserved time-varying coefficients or volatility. Since allowing time variation in coefficients or volatility introduces too many parameters unless restricted, the literature evolved in a way of introducing random processes to time-varying coefficients or volatility to avoid the "overparameterization "problem (Koop and Korobilis (2010)). The randomness in these parameters fits quite well with Bayesian methods because there is no strict distinction between fixed "true "parameters and random samples in the Bayesian tradition.

This chapter will discuss applying Bayesian methods for estimating a time-varying paramter VAR with stochastic volatility using the state space representation of the VAR. Section 2 describes the state space representation and estimation methods for VARs. In particular, each step in the Bayesian estimation procedure of a time-varying parameter VAR with stochastic volatility is explained. Section 3 provides empirical analysis of a time-varying parameter VAR with stochastic volatility using three U.S. macroeconomic variables. We will focus on implications of estimates for the time-varying trend and volatility of each variable during the recent period since the start of the recession of 2007-9. Section 4 concludes.

2 State space representation and Estimation of VARs

2.1 State space representation

Let y_t be an $n \times 1$ vector of observed variables and q the length of lags. A canonical representation of a VAR(q)) with time-invariant parameters and volatility takes the following form.

$$y_t = c_0 + c_1 y_{t-1} + \cdots + c_q y_{t-q} + e_t \ , \ e_t \sim (0, \Sigma_e). \tag{1}$$

Since all the state variables are observed, there is no need to distinguish a state transition equation from a measurement equation in this case. However, if we allow time variation in coefficients $(c_0, c_1, \cdots, c_q)$ or volatility (Σ_e), the model includes some unobserved components as state variables. To estimate these unobserved components based on the observed data, it is useful to distinguish a state transition equation from a measurement equation as in the canonical representation of a state space model. Here are examples of the state space representation of VARs with time-varying coefficients and volatility.

Example 1 (*Time-varying parameter VAR with time-invariant volatility*)

$$y_t = X_t'\theta_t + \epsilon_t \ , \ \epsilon_t \sim \mathcal{N}(0, \Sigma_\epsilon) \ , \ \text{(Measurement Equation)}, \tag{2}$$

$$\theta_t = \theta_{t-1} + v_t \ , \ v_t \sim \mathcal{N}(0, \Sigma_v) \ , \ \text{(State Transition Equation)}. \tag{3}$$

Here X_t includes a constant plus lags of y_t, and θ_t is a vector of VAR parameters. ϵ_t and v_s are assumed to be independent of one another for all t and s. Given the linear and Gaussian state space representation of the above VAR, we can apply the Kalman filter to estimate θ_t conditional on the time series of observed variables y_t. If we further allow a possible correlation between ϵ_t and v_t, the model studied in Cogley and Sargent (2001) belongs to this example.

Example 2 (*VAR with stochastic volatility*)

$$y_t = X_t'\theta + \Sigma_t \epsilon_t \ , \ \epsilon_t \sim \mathcal{N}(0, I_n) \ , \ \Sigma_t = \text{diag}(\sqrt{H_{i,t}}) \ , \ \text{(Measurement Equation)}, \tag{4}$$

$$\ln H_{i,t} = \ln H_{i,t-1} + u_{i,t} \ , \ u_{i,t} \sim \mathcal{N}(0, Q) \ , \ \text{(State Transition Equation)}. \tag{5}$$

Here Σ_t is a diagonal matrix whose diagonal elements are $\sqrt{H_{i,t}}, (i = 1, \cdots, n)$. The measurement equation is a nonlinear function of the unobserved log stochastic volatility $(\ln H_{i,t})$. Hence, the Kalman filter is not applicable in this case. Simulation-based filtering methods are typically used to back out stochastic volatility implied by the observed data. The above model is close to the one studied in Clark (2011), who shows that allowing stochastic volatility improves the real time accuracy of density forecasts out of the VAR model.

Example 3 (*Time-varying parameter VAR with stochastic volatility*)

As emphasized by Sims (2001), ignoring time-varying volatility may overstate the role of time-varying coefficients in explaining structural changes in the dynamics of macroeconomic variables. Adding stochastic volatility to a time-varying parameter VAR will alleviate this concern. The time-varying parameter VAR with stochastic volatility can be described as follows:

$$y_t = X_t'\theta_t + \epsilon_t \ , \ \epsilon_t \sim \mathcal{N}(0, B^{-1}H_tB^{-1'}) \ , \ \text{(Measurement Equation)}, \tag{6}$$

$$\theta_t = \theta_{t-1} + v_t \ , \ v_t \sim \mathcal{N}(0, \Sigma_v) \tag{7}$$

$$\ln H_{i,t} = \ln H_{i,t-1} + u_{i,t} \ , \ u_{i,t} \sim \mathcal{N}(0, Q_i) \ , \ \text{(State Transition Equation)}. \tag{8}$$

Here H_t is a diagonal matrix whose diagonal element is $H_{i,t}$. B^{-1} is a matrix used to identify structural shocks from VAR residuals. If we allow for a correlation between ϵ_t and v_t, this model is the one studied in Cogley and Sargent (2005).[1]

2.2 Estimation of VARs

Without time-varying coefficients or volatility, the VAR can be estimated by equation-by-equation ordinary least squares (OLS) which minimizes the sum of residuals in each equation of the VAR. However, estimating VARs with time-varying coefficients or volatility requires one to use filtering methods to extract information about unobserved states from observed time series. For example, in the time-varying parameter VAR model with time-invariant volatility, we can use the Kalman filter to obtain the estimates of time-varying coefficients conditional on parameters determining the covariance matrix and initial values of coefficients.

Under the frequentist approach, we estimate the covariance matrix and initial values of coefficients first and obtain estimates of time-varying coefficients conditional on the estimated covariance matrix and initial values of coefficients. While conceptually natural, implementing this

[1] If we allow the time variation in the B matrix, the model becomes a time-varying structural VAR in Primiceri (2005). And we can also incorporate the time-varying volatility of v_t too to capture fluctuations in variances of innovations in trend components as in Cogley, Primiceri, and Sargent (2010).

procedure faces several computational issues especially for a high-dimensional model. The likelihood is typically highly nonlinear with respect to parameters to be estimated and maximizing it over a high-dimensional space is computationally challenging.

As emphasized by Primiceri (2005), Bayesian methods can deal efficiently with these types of models using the numerical evaluation of posterior distributions of parameters and unobserved states. The goal of Bayesian inference is to obtain joint posterior distributions of parameters and unobserved states. In many cases such as VARs with time-varying parameters or volatility, these joint distributions are difficult or impossible to characterize analytically. However, distributions of parameters and unobserved states conditional on each other are easier to characterize or simulate. Gibbs sampling, which iteratively draws parameters and unobserved states conditional on each other, provides draws from joint distributions under certain regularity conditions.[2]

As an illustration of Bayesian estimation methods in this context, consider Example 1. Denote z^T be a vector or matrix of variable z_t from $t = 0$ to $t = T$. In this model, unobserved states are time-varying coefficients θ_t and parameters are covariance matrices of VAR residuals and innovations in coefficients $(\Sigma_\epsilon, \Sigma_v)$. Prior distributions for θ_0 and Σ_ϵ can be represented by $p(\theta_0)$ and $p(\Sigma_\epsilon)$. The Bayesian estimation procedure for this model is described as follows:

(***Bayesian Estimation Algorithm for a Homoskedastic Time-varying Parameter VAR***)

Step 1: Initialization

Draw Σ_ϵ from the prior distribution $p(\Sigma_\epsilon)$.

Step 2: Draw VAR coefficients θ^T

The model is a linear and Gaussian state space model. Assuming that $p(\theta_0)$ is Gaussian, the conditional posterior distribution of $p(\theta_t|y^t, \Sigma_\epsilon, \Sigma_v)$ is also Gaussian. A forward recursion using the Kalman filter provides expressions for posterior means and the covariance matrix.

$$\begin{aligned}
p(\theta_t|y^t, \Sigma_\epsilon, \Sigma_v) &= N(\theta_{t|t}, P_{t|t}), \\
P_{t|t-1} &= P_{t-1|t-1} + \Sigma_v, \\
K_t &= P_{t|t-1} X_t (X_t' P_{t|t-1} X_t + \Sigma_\epsilon)^{-1}, \\
\theta_{t|t} &= \theta_{t-1|t-1} + K_t (y_t - X_t' \theta_{t-1|t-1}), \\
P_{t|t} &= P_{t|t-1} - K_t X_t' P_{t|t-1}. \qquad (9)
\end{aligned}$$

[2]See Lancaster (2004, Chap. 4) for necessary conditions.

Starting from $\theta_{T|T}$ and $P_{T|T}$, we can run the Kalman filter backward to characterize posterior distributions of $p(\theta^T|y^T, \Sigma_\epsilon, \Sigma_v)$.

$$\begin{aligned} p(\theta_t|\theta_{t-1}, y^T, \Sigma_\epsilon, \Sigma_v) &= N(\theta_{t|t+1}, P_{t|t+1}), \\ \theta_{t|t+1} &= \theta_{t|t} + P_{t|t}P^{-1}_{t+1|t}(\theta_{t+1} - \theta_{t|t}), \\ P_{t|t+1} &= P_{t|t} - P_{t|t}P^{-1}_{t+1|t}P_{t|t}. \end{aligned} \tag{10}$$

We can generate a random trajectory for θ^T using the backward recursion starting with a draw of θ^T from $\mathcal{N}(\theta_{T|T}, P_{T|T})$ as suggested by Carter and Kohn (1994).

Step 3: Draw covariance matrix parameters for VAR coefficients Σ_v
Conditional on a realization for θ^T, innovations in VAR coefficients v_t are observable. Assuming the inverse-Wishart prior for Σ_v with scale parameter $\overline{\Sigma_v}$ and degree of freedom T_{v0}, the posterior is also inverse-Wishart.[3]

$$\begin{aligned} p(\Sigma_v|y^T, \theta^T) &= IW(\Sigma^{-1}_{v,1}, T_{v,1}), \\ \Sigma_{v,1} &= \overline{\Sigma_v} + \sum_{t=1}^{T} v_t v_t' \ , \ T_{v,1} = T_{v,0} + T. \end{aligned} \tag{11}$$

Step 4: Draw covariance matrix parameters for VAR residuals Σ_ϵ
Conditional on a realization for θ^T, VAR residuals ϵ_t are observable. Assuming the inverse-Wishart prior for Σ_ϵ with scale parameter $\overline{\Sigma_\epsilon}$ and degree of freedom $T_{\epsilon,0}$, the posterior is also inverse-Wishart.

$$\begin{aligned} p(\Sigma_\epsilon|y^T, \theta^T) &= IW(\Sigma^{-1}_{\epsilon,1}, T_{\epsilon,1}), \\ \Sigma_{\epsilon,1} &= \overline{\Sigma_\epsilon} + \sum_{t=1}^{T} \epsilon_t \epsilon_t' \ , \ T_{\epsilon,1} = T_{\epsilon,0} + T. \end{aligned} \tag{12}$$

Step 5: Posterior Inference
Go back to step 1 and generate new draws of θ^T, Σ_v, and Σ_ϵ. Repeat this $M_0 + M_1$ times and discard the initial M_0 draws. Use the remaining M_1 draws for posterior inference. Since each draw is generated conditional on the previous draw, posterior draws are generally autocorrelated. To

[3] Notations here closely follow those in the appendix of Cogley and Sargent (2005).

reduce the autocorrelation, we can thin out posterior draws by selecting every 20th draw from M_1 draws, for example.

Although posteior draws are obtained from conditional distributions, their empirical distributions approximate the following joint posterior distribution $p(\theta^T, \Sigma_v, \Sigma_\epsilon | y^T)$. Hence, integrating out uncertainties about other components of the model is trivial. For instance, if we are interested in the median estimate of θ^T, which integrates out uncertainties of Σ_v and Σ_ϵ, we can simply use the median value of M_1 draws of θ^T.

In the above example, conditional distributions of parameters and unobserved states are known. However, in models with stochastic volatility such as Example 2 and Example 3, the conditional posterior distribution of volatility is known up to a constant. Without knowing the constant, we can directly sample from the conditional posterior distribution for stochastic volatility. Instead, we can use Metropolis-Hastings algorithm following Jacquier, Polson, and Rossi (1994) to generate posterior draws for stochastic volatility.

As an illustration, consider Example 3. Posterior simulation for time-varying coefficients θ^T and Σ_v are essentially the same as before. Below, I will describe steps to generate posterior draws for $H_{i,t}$ and B conditional on θ^T, y^T, and Σ_v.

(***Drawing stochastic volatility*** H_t ***and covariance parameters*** B)

Step 1 Given y^T, θ^T, we can generate a new draw for B. Notice the following relationship between VAR residuals ϵ_t and structural shocks u_t.

$$B\epsilon_t = u_t. \tag{13}$$

Conditional on y^T and θ^T, ϵ_t is observable. Since B governs only covariance structures among different shocks, $\frac{n(n+1)}{2}$ elements of the matrix are restricted. For example, if B is the following 2×2 matrix,

$$B = \begin{pmatrix} 1 & 0 \\ B_{21} & 0 \end{pmatrix},$$

with $B_{21} \sim \mathcal{N}(\overline{B}_{21}, V_{21})$, the relation between ϵ_t and u_t implies the following transformed regressions.

$$\begin{aligned} \epsilon_{1t} &= u_{1t} \\ (H_{2t}^{-.5}\epsilon_{2t}) &= B_{21}(-H_{2t}^{-.5}\epsilon_{1t}) + (H_{2t}^{-.5}u_{2t}). \end{aligned} \tag{14}$$

As explained by Cogley and Sargent (2005), the above regressions imply the normal posterior for B_{21}.

$$B_{21}|y^T, H^T, \theta^T \sim \mathcal{N}(\hat{B}_{21}, \hat{V}_{21}) \ , \ \hat{V}_{21} = (V_{21}^{-1} + \sum(\frac{\epsilon_{1t}^2}{H_{2t}}))^{-1} \ , \ \hat{B}_{21} = \hat{V}_{21}(V_{21}^{-1}\overline{B}_{21} - \sum(\frac{\epsilon_{1t}\epsilon_{2t}}{H_{2t}})). \tag{15}$$

Step 2 Conditional on ϵ_t, we can write down the following state representation for H_t.

$$\sum_{j=1}^{n} B_{ij}\epsilon_{jt} = \sqrt{H_{it}}w_{it} \ , \ w_{it} \sim i.i.d.\mathcal{N}(0,1), \tag{16}$$

$$\ln H_{i,t} = \ln H_{i,t-1} + u_{i,t} \ , \ u_{i,t} \sim \mathcal{N}(0, Q_i). \tag{17}$$

The above system is not linear and Gaussian with respect to H_{it}. The conditional posterior density of H_{it} is difficult to characterize analytically but known up to a constant.

$$\begin{aligned} p(H_{it}|H_{i,t-1}, H_{i,t+1}, y^T, \theta^T, B, Q_i) &\propto p(u_{it}|H_{it}, B)p(H_{it}|H_{i,t-1})p(H_{i,t+1}|H_{it}), \\ &\propto H_{it}^{-1.5} exp(-0.5\frac{u_{it}^2}{H_{it}}) exp(-0.5\frac{(\ln H_{it} - \mu_{it})^2}{0.5Q_i}), \\ \mu_{it} &= 0.5(\ln H_{i,t-1} + \ln H_{i,t+1}). \end{aligned} \tag{18}$$

We can use the Metropolis-Hastings algorithm which draws H_{it} from a certain proposal density $q(H_{it})$.[4] Each mth draw is accepted with probability α_m,

$$\alpha_m = \frac{p(H_{it}^m|H_{i,t-1}^m, H_{i,t+1}^{m-1}, y^T, \theta^T, B, Q_i)q(H_{i,t}^{m-1})}{p(H_{it}^{m-1}|H_{i,t-1}^m, H_{i,t+1}^{m-1}, y^T, \theta^T, B, Q_i)q(H_{i,t}^m)}. \tag{19}$$

As shown by Jacquier, Polson, and Rossi (1994), this sampling scheme generates posterior draws for H_{it}.

[4] One example of such a proposal density is $\mathcal{N}(\mu_{it}, 0.5Q_i)$.

3 Application: A Time-varying Parameter VAR with Stochastic Volatility for the three U.S. Macroeconomic Variables

As an application of state space modelling, we estimate a time-varying parameter VAR with stochastic volatility for U.S. macroeconomic time series consisting of inflation, the unemployment rate and the long-term interest rate.[5] The model is close to Cogley and Sargent (2005) but there are two main differences. First, we shut down the correlation VAR residuals and innovations in time-varying parameter transition equations. Second, we use the long-term interest rate rather than the short-term interest rate to cover overall monetary policy stance at the recent zero lower bound period. The estimated model can be casted into the following state space representation like **Example 3** in the previous section.

$$\begin{aligned} y_t &= \theta_{0,t} + y_{t-1}\theta_{1,t} + y_{t-2}\theta_{2,t} + \epsilon_t \ , \ \epsilon_t \sim \mathcal{N}(0, B^{-1}H_t B^{-1\prime}), \\ \theta_t &= \theta_{t-1} + v_t \ , \ \theta_t = [\theta_{0,t}{}', \theta_{1,t}{}', \theta_{2,t}{}']' \ , \ v_t \sim \mathcal{N}(0, \Sigma_v), \\ \ln H_{i,t} &= \ln H_{i,t-1} + u_{i,t} \ , \ u_{i,t} \sim \mathcal{N}(0, Q_i) \ , \ (i = 1, 2, 3). \end{aligned} \tag{20}$$

y_t contains three variables in the order of the 10-year yield, core PCE inflation, and the civilian unemployment rate. The sample period is from 1960:Q1 to 2011:Q4. We assume that B is a lower triangular matrix whose diagonal elements are all equal to 1.

3.1 Priors

Priors are set in the same way as Cogley and Sargent (2005), using pre-sample data information from 1953:Q2 to 1959:Q4.[6]

First, we estimate seemingly unrelated regressions for the pre-sample data and use the point estimate of coefficients as the prior mean for θ_0 and its asymptotic variance $\overline{P}$ as the prior variance. Second, we use an inverse-Wishart distribution as the prior for Σ_v with degree of freedom $T_0 = 22$ and scale matrix $\overline{\Sigma_v} = T_0 \times 0.001 \times \overline{P}$. Third, the prior distribution of the log of the initial volatility is set to the normal distribution whose mean is equal to the variance of regression residuals using

[5] Doh (2011) estimates the same model with shorter sample data and focuses on the time-varying relationship between inflation and unemployment.

[6] For pre-sample data, we use total PCE inflation because core PCE inflation is not available for this period.

the pre-sample data. The prior variance is set to 10. Fourth, the prior distributions of elements in B are normal with the mean equal to 0 and the covariance matrix equal to $10000 \times I_3$. Finally, the prior for the variance of the innovation to volatility process is inverse gamma with the scale parameter equal to 0.01^2 and the degree of freedom parameter equal to 1.

3.2 Posterior Simulation

We generate 100,000 posterior draws and discard the first 50,000 draws. Among the remaining 50,000 draws, I use every 20th draw to compute posterior moments. Following Cogley and Sargent (2005), we throw away draws implying the non-stationarity of the VAR. Hence, if θ^T contains coefficients which indicate the non-stationarity of the VAR at any point of time, we redraw θ^T until the stationarity is ensured all the time.

Consider a companion VAR(1) for $[y_t', y_{t-1}']'$. Technically speaking, stationarity is guaranteed if all the eigenvalues of $A_t = \begin{pmatrix} \theta_{1,t} & \theta_{2,t} \\ I_3 & 0 \end{pmatrix}$ are inside the unit circle. The truncation is particulary useful when we back out time-varying trend components from estimated coefficients. When we use the companion form for long-horizon forecasts, the stochastic trend in $[y_t', y_{t-1}']'$ can be approximated as $(I - A_t)^{-1}[\theta_{0,t}', 0]'$. Below, we will use this approximation to obtain posterior estimates of time-varying trends in y_t.

3.3 Posterior estimates of time-varying trends and volatility

Over the last fifty years, the U.S. economy has shown substantial changes. In particular, there is considerable evidence that trend inflation and volatility of inflation rose during the mid 1970s and the early 1980s but then declined after the Volcker disinflation.[7] Also, the decline in the volatility of inflation is one primary factor for explaining the decline in the term premium of long-term government bonds since the late 1980s (Wright (2011)). On the other hand, economic slack seems to be less important in predicting inflation since 1984.[8] In addition, the volatility of real activity declined since the mid 1980s.[9]

Most papers on these issues rely on data before the most recent recession that started in

[7] For example, see Cogley, Primiceri, and Sargent (2010) and papers cited there.

[8] See Doh (2011) and papers discussed there.

[9] See Canova and Gambetti (2009) and papers cited there.

late 2007. The severity of the recession and the unprecedented policy actions including keeping the short-term interest rate at the effective zero lower bound and implementing large-scale asset purchases raised a question about the robustness of the above-mentioned changes.

Our time-varying parameter VAR model with stochastic volatility can shed light on this question. First of all, we can investigate if the recession and the subsequent policy responses affected mainly trend components or cyclical components of the three macroeconomic variables. Impacts on cyclical components are expected to be temporary while those on trend components are supposed to be more persistent. Second, we can do a similar exercise for the volatility of the three macroeconomic variables. For instance, we can compute the short-run and the long-run volatility of the three variables in the VAR and see if there might be shifts during the recent period.

Our posterior estimates of time-varying trends in Figure 1 suggest that trends in nominal variables such as inflation and the long-term interest rate were little affected by the recent episode while the trend unemployment rate was affected more substantially. This result is interesting because the level of all the variables moved significantly during the same period as shown in Figure 2. For the inflation rate, movements in the trend component explain about 9 percent of the overall movements in the level of the inflation rate.[10] The relative contribution of the trend component further declines to about 6 percent for the nominal ten-year bond yield. In contrast, the movement in the trend unemployment rate explains more than 15 percent of the overall movement in the unemployment rate for the same period.

These differences in the relative contribution of time-varying trends across variable suggest that it will take a longer time for the unemployment rate to return to the pre-recession level than other variables. However, it is possible that the relatively small role of trend component volatility was driven by our assumption of constant volatility of innovations in time-varying coefficients. To check the robustness of our finding, we allowed for the time-varying volatility for innovations in θ_t in an alternative specification. Even is this version of the model, we got essentially the same relative contribution of trend components during the recent period.[11]

We can apply the similar trend-cycle decomposition for volatility estimates, too. Following Cogley, Primiceri, and Sargent (2010), we approximate the unconditional variance of $[y_t', y_{t-1}']'$ by

[10]This calculation is based on comparing the standard deviation of each variable during the relevant period.

[11]The drawback of this generalization of time-varying volatility is that so many volatility estimates become explosive during the mid 1970s, casting doubts on the convergence property of the model estimates. For the model without stochastic volatility for innovations in θ_t, we do not observe such a convergence issue.

$$\sum_{h=0}^{\infty}(A_t)^h B^{-1} H_t B^{-1\prime}((A_t)^h)'. \quad (21)$$

This unconditional variance is dominated by slowly moving trend components of volatility estimates while H_t is mainly based on the short-run movements of volatility estimates. The volatility of residuals went up for all the variables as shown in Figure 3. In addition, the unconditional volatility of the unemployment rate moved up more noticeably during the recent period to the historical peak level as shown in Figure 4. The finding suggests that the increase in volatility since the recession may not be driven by a common factor affecting the entire economy. This interpretation is in line with the observation in Clark (2009) that the recent increase in volatility is concentrated in certain sectors of the economy (goods production and investment but not services components, total inflation but not core).

Overall, our posterior analysis indicates that the trend and volatility of the unemployment rate have experienced substantial changes during the recent episode while core inflation and the nominal long-term interest rate have been relatively immune from these changes. Analyzing causes of different responses across variables may require a more structural model of the economy built on decisions of agents. Our analysis can be a starting point for such a project.

4 Conclusion

VARs are widely used in macroeconomics and finance to describe the historical dynamics of multiple time series. When the VAR is extended to incorporate time-varying coefficients or volatility to capture structural shifts in the economy over time, the state space representation is necessary for the estimation. Applying the Kalman filter in the state space representation of a time-varying parameter VAR provides estimates of unobserved time-varying coefficients that we are interested in. Also, we can obtain estimates of time-varying volatility by applying the simulation-based filtering method to the state space representation of the volatility process.

We illustrate the value of applying the state space representation to the time-varying parameter VAR with stochastic volatility by estimating such a model with the three U.S. macro variables. Our empirical analysis suggests that the recession of 2007-9 was driven by a particulary bad shock to the unemployment rate which increased the trend and volatility of the unemployment rate substantially. In contrast, nominal variables such as the core PCE inflation rate and the ten-year Treasury bond yield have exhibited relatively less noticeable movements in terms of their trend and volatility. Further identifying underlying causes of unemployment dynamics may requires us to go beyond the small scale time-varying parameter VAR model that we are considering in this chapter.

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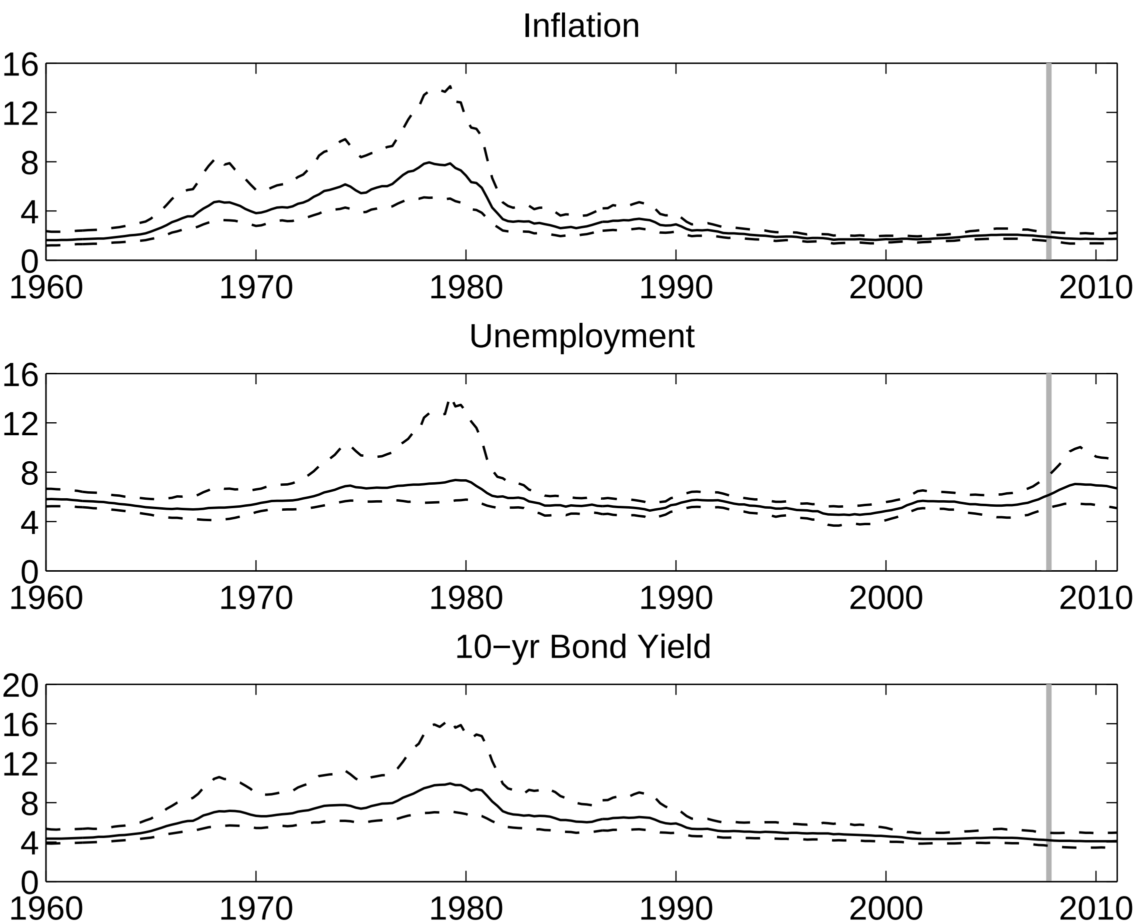
Figure 1: Time-varying trend



The solid line stands for posterior median estimates and dashed lines for estimates of the 70 percent highest posterior density regions. The vertical bar indicates the fourth quarter of 2007 when the recession started.

Figure 2: Data

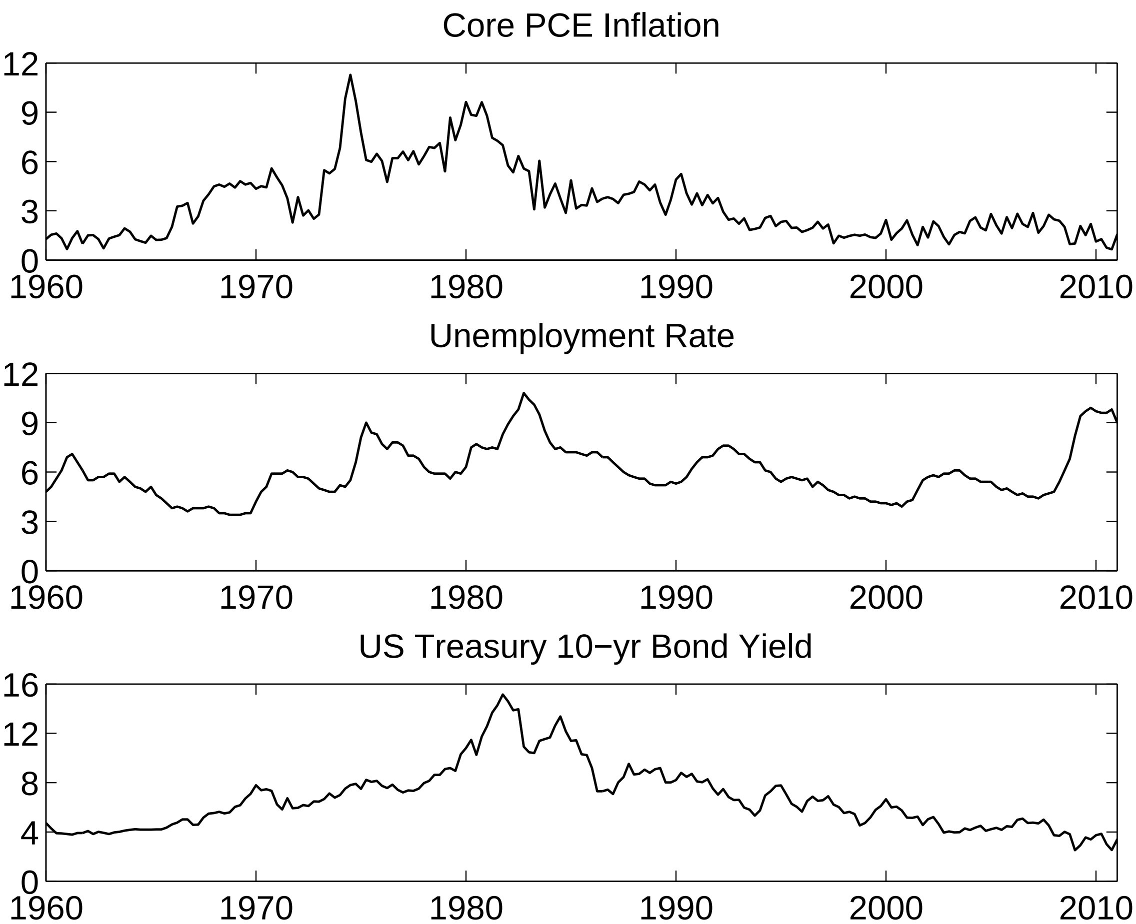
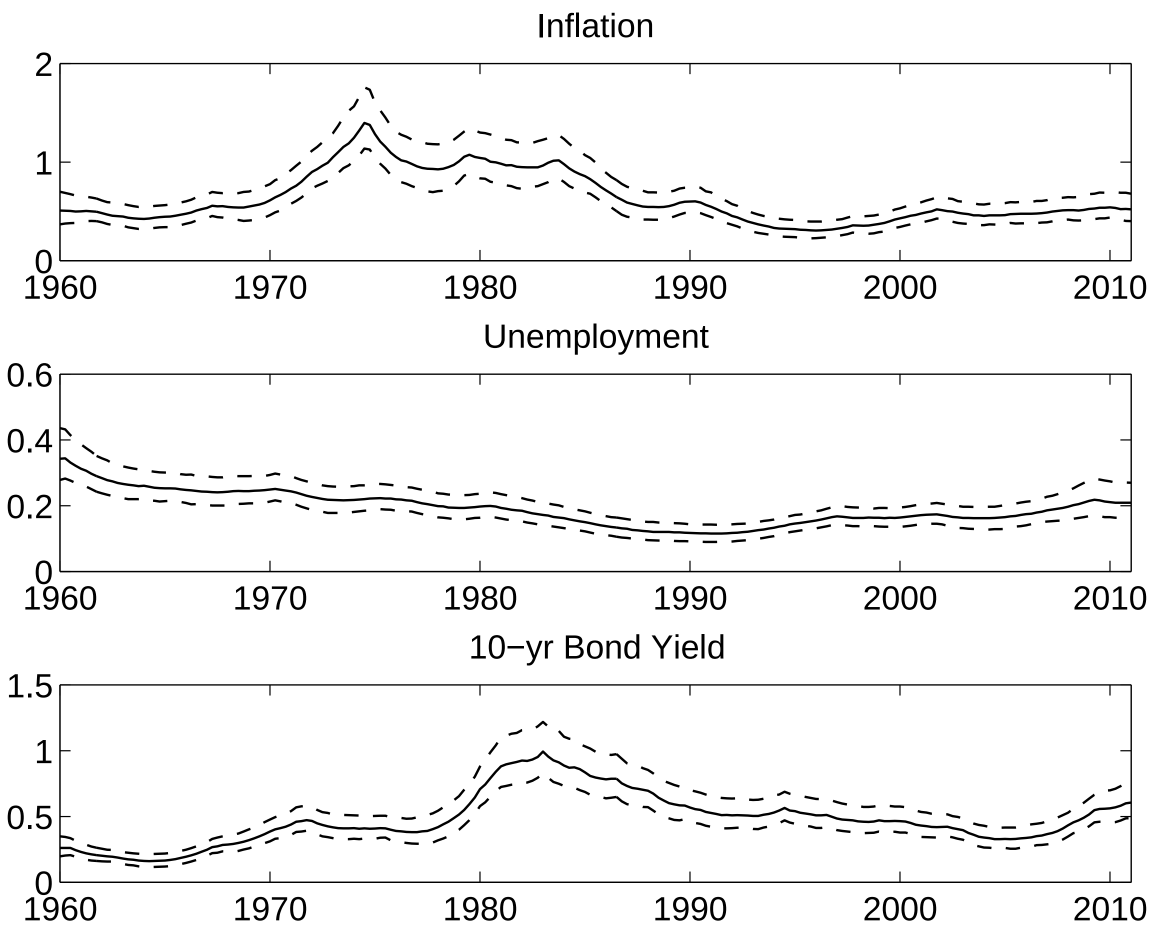
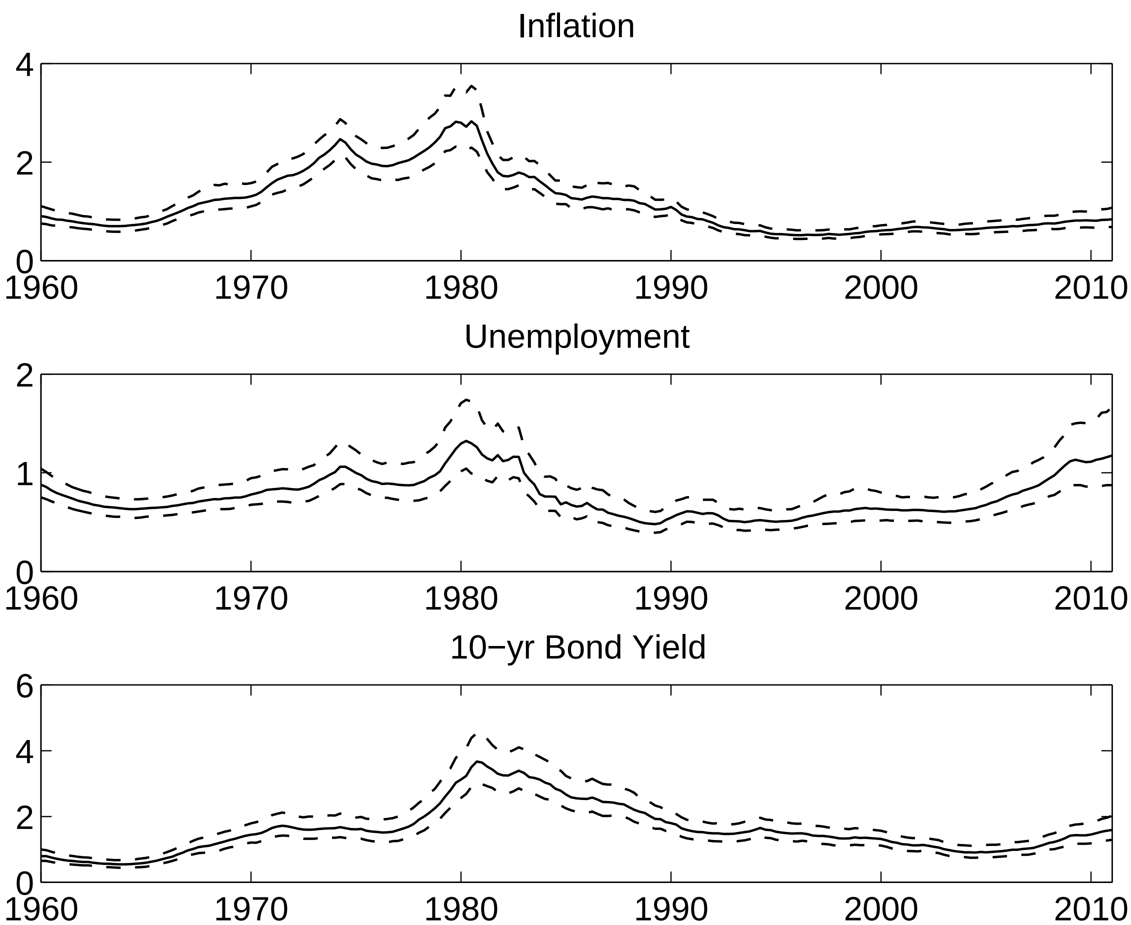


Figure 3: Time-varying volatility of residuals



The solid line stands for posterior median estimates and dashed lines for estimates of the 70 percent highest posterior density regions.

Figure 4: Time-varying unconditional volatility



The solid line stands for posterior median estimates and dashed lines for estimates of the 70 percent highest posterior density regions.