VECTOR RATIONAL ERROR CORRECTION

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Abstract: Systems of forward-looking linear decision rules can be formulated as vector "rational" error correction models. The closed-form solution of the restricted error corrections is derived, and a full-information estimator is suggested. The error correction format indicates that the assumptions of convex adjustment costs and rational expectations impose different types of a priori restrictions on the dynamic structure of the error corrections. An empirical model of the producer decision rule for capital investment illustrates that the data rejects dynamic restrictions imposed by a standard model of adjustment costs but supports a more general description of convex frictions.

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1. Introduction

Policy modeling in macroeconomics is complicated by theory, fit, and flexibility requirements. An influential theoretical standard is that models should provide an explicit template of rational decision making by economic agents. Typical data-based objectives for estimated relationships are to explain historical fluctuations in macroeconomic aggregates and to forecast future movements. A third model requirement is to select flexible specifications of features where economic priors are weak and data are scant, such as assumptions on the information and dynamic adjustment constraints of firms and households.

Macroeconomic modelers are forced to make significant concessions in attempting to simultaneously address these competing requirements. Thus, it is not surprising that different modeling strategies have evolved that better address one requirement more than others. Between the boundaries of data-based, atheoretic time series models, such as the VAR specifications developed by Sims (1980), and of theory-based simulation models, discussed by Kydland and Prescott (1996), are many hybrid modeling strategies. A defining characteristic of these hybrid approaches is the introduction of sufficient prior restrictions into data-based specifications to enable coherent economic analysis of a particular issue. Examples of such hybrid model methodologies range from the use of long-run priors in error correction models, discussed in Hendry (1995), to explicit assumptions regarding the timing of responses to information in structural VARs, reviewed in Watson (1994).

This paper develops a hybrid modeling framework that provides a tractable bridge between data-based and theory-based modeling in the case of linear decision rules. Given that a system of linear decision rules for I(1) instruments is isomorphic to a restricted class of vector error correction (VEC) models, the purpose of this paper is to examine the error correction equations implied by standard formulations of decision rules. The paper indicates that a subset of a priori dynamic restrictions, imposed by many implementations of intertemporal optimization, is generally rejected by error correction models of macroeconomic aggregates.

Discussion is organized as follows. Using the example of a single decision variable, section 2 contrasts the role of a priori restrictions in conventional error correction models and in the standard linear decision rules implemented in many existing macroeconomic models. The common specification where convex adjustment costs are applied only to the *level* of decision variables is suggested as a potential source of dynamic misspecification and of undesirable properties in representative empirical estimates of decision rules in macroeconomics.

A modeling framework for a more general description of frictions is presented in section 3. A system of rational decision rules is derived for a vector of decision variables using the extension of convex frictions to polynomial lag functions of decision variables suggested by Tinsley (1993). For difference-stationary variables, the resulting decision rules comprise a system of vector "rational" error correction equations (VREC) with cross-coefficient restrictions due to convex frictions and cross-equation restrictions due to the assumption of rational expectations. These restrictions are presented in analytic closed-form using a solution method based on lead and lag companion systems.

Section 4, supplemented by subsections in the appendix, discusses full-information maximum likelihood estimation of the vector rational error correction system. A tractable two-step procedure is suggested, along with corrections of standard errors for the "generated regressor" bias that is associated with two-step estimators. For cases where the forcing terms of Euler equations are influenced by feedback from decision variables, model-consistent expectations are imposed by mapping the decision rules into the forecasting system for the forcing terms.

Section 5 provides an empirical example. The methodology is applied to a model of producers' investment in capital equipment with time-to-build constraints and convex costs of adjusting the rate of equipment installations. Section 6 concludes.

2. Atheoretic and A Priori Descriptions of Dynamic Adjustment

From seminal work by Tinbergen in the 1930s to the present, the introduction of priors from economic theory into linear dynamic models remains somewhat of an art form, subject to cycles in theoretical fashion as well as trends in analytical and computational techniques.

In contrast to the "measurement without theory" trend and cycle decompositions of prewar business cycle analysts, postwar researchers at the Cowles Commission indicated that economic theory can be a source of exclusion restrictions in constructing empirical models. Decision instruments are related to subsets of candidate explanatory variables by introducing a priori zero coefficient restrictions on both contemporaneous and lagged contributions of selected regressors. However, because much of the inherited theory in macroeconomics is viewed largely a source of static priors about equilibrium relationships, the assessment by Sims (1980) that dynamic exclusion restrictions were "incredible" stimulated widespread use of atheoretic VARs.

In recent years, there has been a revival of interest in a priori restrictions to develop structural VARs where a priori zero restrictions are placed on the covariances of model disturbances, as reviewed by Watson (1994) and Canova (1995). In many instances, these restrictions are equivalent to non-dynamic versions of exclusion conditions. Examples include a priori zero restrictions on contemporaneous coefficients, such as the restrictions of current-period responses to shocks in Bernanke and Mihov (1995) and Christiano, Eichenbaum, and Evans (1996), and a priori zero restrictions on selected steady-state coefficients, such as the long-run neutrality assumption in Blanchard and Quah (1989).

In contrast to the structural VAR literature on static restrictions, this section discusses two theory-based sources of dynamic restrictions. These restrictions are less frequently implemented in estimated macroeconomic models due to an extensive literature, reviewed by Ericsson and Irons (1995), that suggests these restrictions are not supported by macroeconomic data. These include restrictions associated with intertemporal optimization under convex frictions and restrictions imposed by an explicit forecast model of agent expectations. Because agent expectations are generally unobserved, the assumption of rational expectations (RE) is often invoked to equate the agent forecast model with the data generating mechanism of the explanatory forcing variables. Although both sources of restrictions are collectively referenced as "RE overidentifying restrictions," this grouping masks the different types of restrictions contributed by these assumptions and, thus, the source of empirical rejections.

The RE forecast assumption imposes cross-equation restrictions on dynamic coefficients in the agent decision rule and dynamic coefficients in the agent forecast model of forcing variables. By contrast, the intertemporal optimization assumption imposes lead and lag cross-coefficient restrictions that enforce similarities between the relative importance of recent and older shocks and the relative importance of nearby and distant anticipations. Discussion will indicate that conventional specifications of frictions imply a priori zero restrictions on certain dynamic coefficients when the decision rule is reformulated as a restricted error correction, and these zero restrictions are almost always rejected by macroeconomic aggregates. For simplicity, only the case of a single decision variable is considered.

2.1 Conventional error correction

To establish notation, the decision variable of interest is y_t, the set of k explanatory variables is $x_t' \equiv [x_{1,t}, x_{2,t}, \ldots, x_{k,t}]$, and a q-order VAR references q lags of each regressor in the $kq \times 1$ information vector of agents containing the explanatory variables, $z_{x,t-1}' = [x_{t-1}', x_{t-2}', \ldots, x_{t-q}']$. Thus, in the case of an unrestricted VAR, the equation for y_t is represented by

$$
\begin{aligned}
E_{t-1}y_t &= \mu_y + a_0(L)y_{t-1} + \sum_{j=1}^{k} a_j(L)x_{j,t-1}, \\
&= \mu_y + a_0(L)y_{t-1} + a'z_{x,t-1},
\end{aligned} \tag{1}
$$

where E_{t-1} denotes the expectation conditioned on agent information available at the end of period $t-1$, $a_j(L)$ denotes a $(q-1)$-order scalar polynomial in the lag operator, $L^i x_t = x_{t-i}$, and a is the unrestricted $kq \times 1$ coefficient vector of the information vector, $z_{x,t-1}$.

For stationary series, the equilibrium target for the decision variable implied by equation (1) is

$$
\begin{aligned}
y_t^* &= (1 - a_0(1))^{-1}[\mu_y + \sum_{j=1}^{k} a_j(1)x_{j,t}], \\
&= c_0 + \sum_{j=1}^{k} c_j x_{j,t}.
\end{aligned} \tag{2}
$$

In modeling the levels of variables, the assumption of difference-stationarity, I(1), is not rejected for many macroeconomic aggregates. In these instances, the representation in the second line of equation (2) will denote the equilibrium path of the decision target established by cointegration analysis.

In the case of a cointegrating target path, equation (1) can be rewritten as an error correction description of the adjustment to the equilibrium path.

$$
\begin{aligned}
E_{t-1}\Delta y_t &= \mu - \gamma(y_{t-1} - y_{t-1}^*) + b_0(L)\Delta y_{t-1} + \sum_{j=1}^{k} b_j(L)\Delta x_{j,t-1}, \\
&= \mu - \gamma(y_{t-1} - y_{t-1}^*) + b_0(L)\Delta y_{t-1} + b' R_b z_{x,t-1}.
\end{aligned} \tag{3}
$$

The coefficients of the first line in (3) are unrestricted. To enable later comparisons with error correction formulations of decision rules, the second line references the information vector, $z_{x,t-1}$. This more compact formulation requires a restriction matrix, R_b, to account for the first-difference formats of the corresponding regressors in the first line.[1]

As noted by Hendry (1995), many macroeconomic models estimated in the United Kingdom are error correction systems. An attractive feature of the error correction

[1]The restriction matrix imposes k zero sum restrictions on the coefficients in b. Given the ordering of variables in the information vector, $R_b \equiv I_{kq} - (Q \otimes I_k)$, where each element of the $q \times q$ matrix, Q, is q^{-1}. Equivalently, R_b can be interpreted as transforming the regressors into deviations from q-period averages.

specification is that additional "static" or equilibrium priors from economic theory, such as long-run homogeneity of prices or Cobb-Douglas production, can be introduced into the cointegrating relation, equation (2). Indeed, use of prior information to identify long-run outcomes is very useful for macroeconomists confronted by short samples. Also, as shown by Horvath and Watson (1992), the power of cointegration tests is improved for cointegrations with known parameters.

As with unrestricted VAR models, error correction models generally provide good empirical fits of macroeconomic aggregates. While error correction models often satisfy data-based objectives of modelers, the absence of explicit distinctions between dynamics due to adjustment costs and to revisions of agent forecasts precludes theory-based interpretations of economic events based on rational agent responses to "news." In this sense, error correction equations continue to inherit many of the theoretical limitations of traditional dynamic macroeconomic models developed in the 1960s. As discussed by Lucas (1976) and Sargent (1981), models that weld together response dynamics and forecast dynamics into invariant lag structures are less well-suited for policy analysis than models that explicitly represent behavior as a consequence of rational planning by optimizing firms and households.

2.2 Rational error correction

Apart from signal extraction filters and scheduling or delivery lags, the main rationalization of time series dynamics in macroeconomic models is that intertemporal optimizations of utility or profits are subject to frictions on the adjustments of decision variables. In turn, for the case of linear decision rules, there are two interpretations of frictions. One, advanced by Calvo (1983), is that each agent is subjected to a distribution schedule of random delays in adjustment, so that an agent's setting of a decision variable in a given period is, in effect, a weighted average of desired target settings over the expected interval between allowable resets. The other, often attributed in macroeconomics to Eisner and Strotz (1963) in the case of fixed targets, is that movements of the decision variable are subject to quadratic adjustment costs, and these strictly convex frictions induce gradual adjustments toward the desired target setting.[2] In the simple cases usually implemented in macroeconomic models, either the assumption of a *geometric* distribution of random delays (constant hazard rate) or the assumption of a quadratic cost in adjusting the *level* of the decision variable produce observationally equivalent decision rules, Rotemberg (1996).

[2]Additional references and discussion of issues in dynamic model interpretations of decision rules with dynamic forcing terms are found in Tinsley (1970, 1971).

In the latter case, the representative agent selects a current setting for the decision instrument that minimizes the expectation of a multiperiod criterion, $y_t = \text{argmin}(E_{t-1}\{\mathbf{C_t}\})$. Letting y_t^* denote the equilibrium settings that maximize profits or utility in the absence of frictions, the standard quadratic tracking criterion trades off deviations from equilibrium settings against the cost of adjusting the current *level* of the decision variable.

$$\mathbf{C_t} = \sum_{i=0}^{\infty} B^i[(y_{t+i} - y_{t+i}^*)^2 + c_1(y_{t+i} - y_{t+i-1})^2], \tag{4}$$

where c_1 denotes the cost of adjustment relative to the cost of disequilibrium, and B is a fixed discount factor, $0 < B < 1$. In the case of a trending equilibrium path, adjustment cost terms may be reformulated as costly deviations around expected trend growth to ensure that planned settings converge to the target equilibrium path.

With adjustment costs imposed only on changes in the level of the decision variable, the Euler equation that defines the optimal current setting, y_t, is second-order,

$$E_{t-1}\{\lambda(BF)\lambda(L)y_t - \lambda(B)\lambda(1)y_t^*\} = 0, \tag{5}$$

where $\lambda(BF) \equiv 1 - \lambda_1 BF$ and $\lambda(L) \equiv 1 - \lambda_1 L$ are scalar polynomials in the lead, F, and lag, L, operators, and λ_1 is the fractional solution of the equation, $c_1 = \lambda_1/[(1-\lambda_1)(1-\lambda_1 B)]$.

Most current data-based macroeconomic models are constructed using estimated Euler equations of the form in equation (5)[3] A less common approach is full-information maximum likelihood estimation of macroeconomic decision rules where specification of the data generating process of the forcing term, y^*, is required. In early examples, such as the seminal study by Sargent (1978), autoregressive models of the forcing term were used. To facilitate term-by-term contrasts of the dynamic structures of optimal decision rules and conventional error corrections, we will use the more general specification that the k determinants of the forcing term are generated by a q-order VAR, $E_{t-1}x_t = \sum_{j=1}^{q} A_j x_{t-j}$. Under rational expectations, this model matches the agent forecast model. Recast into a first-order companion form, the forecast model is

$$E_{t-1}y_{t+i}^* \quad = \quad \iota_*' H^{1+i} z_{x,t-1}. \tag{6}$$

[3]Second-order Euler equations have been used extensively in macroeconomic policy models estimated in North America and elsewhere. Summary descriptions of several such models are provided in Bryant *et al.* (1993). In most instances, these are directly estimated by GMM-IV, although recent policy models also rely on calibration of simulation properties, such as staff models constructed for the Bank of Canada, Poloz *et al.* (1994), the Federal Reserve Board, Levin *et al.* (1997), and the Reserve Bank of New Zealand, Black *et al.* (1997),

As before, $z_{x,t-1}$ denotes the $kq \times 1$ information vector of agents, ι_* is a $kq \times 1$ selector vector for y_t^*, containing the relevant cointegration coefficients defined in equation (2), and H is a $kq \times kq$ top-row companion matrix of the forecast model

$$H = \begin{bmatrix} A_1 & A_2 & \dots & A_q \\ & I_{kq-q} & & 0 \end{bmatrix}.$$

.

As shown in Tinsley (1970), the optimal decision rule that satisfies the Euler equation and the relevant endpoint (initial and transversality) conditions implies partial adjustment to a discounted average of the forward equilibrium path. The decision rule solution can be obtained by multiplying equation (5) by the inverse of the lead polynomial, $\lambda(BF)^{-1}$, expanding the forward summation of discounted forward targets, and substituting in forecasts of the expected target over the planning horizon from the model representing agent expectations, equation (6). Following Tinsley (1993), the error correction format of the optimal decision rule can be shown to be

$$E_{t-1}\Delta y_t \quad = \quad -\lambda(1)(y_{t-1} - y_{t-1}^*) + h_*' z_{x,t-1}, \qquad (7)$$

where the $kq \times 1$ coefficient vector of the agents' information set, $z_{x,t-1}$, is defined by $h_* \equiv \lambda(1)[H' - I_{kq}][I_{kq} - \lambda_1 BH']^{-1}\iota_*$.

A term-by-term comparison of the conventional error correction in equation (3) with the "rational" error correction in equation (7) indicates two differences in a priori restrictions on the dynamic formats:

First, conditional on estimates of λ_1 from the coefficient of the cointegrating discrepancy, $y_{t-1} - y_{t-1}^*$, of the fixed discount factor, B, and of H from estimation of the forecast model, the $kq \times 1$ coefficient vector of the information vector, h_*, is completely determined. In other words, there are $kq - k$ free parameters in the $b'R_b$ coefficient vector of the conventional error correction, equation (3), but there are no free parameters in the h_* coefficient vector of the decision rule, equation (7).

Second, turning to the remaining terms, note that the q lags of the decision variable in the conventional error correction, equation (3), are now replaced in equation (7) by the single lag of the decision variable indicated in the lagged cointegrating discrepancy, $y_{t-1} - y_{t-1}^*$. Thus, the decision rule formulation, equation (7), imposes $q-1$ additional zero coefficient restrictions on lags of the decision variable. Given estimates of the companion matrix, H, of the forecast model and the discount factor, B, the only free parameter in the rational error correction, equation (7), is the adjustment cost parameter, λ_1.

One interpretation of the often disappointing empirical performances of conventional two-root decision rules, including rejections of rational expectation restrictions, is simply that expectations of actual agents may not be formed under conditions required for rational expectations, such as symmetric access to full system information by all agents. However, the limited dynamic specifications illustrated in equation (7) suggest another contributing factor—the assumption that adjustment costs smooth only changes in the *level* of the decision variable. This imposes $q-1$ additional zero restrictions on the transfer function associated with y. To explore consequences of relaxing this arbitrary dynamic restriction, Tinsley (1993) suggests a generalization of frictions where costs of adjusting decision variables are a function of a higher order polynomial in the lag operator or, equivalently, that the order of the relevant linear Euler equation is higher than the two-root example usually implemented in macroeconomics, such as equation (5).

Two general interpretations of polynomial frictions and the associated higher-order Euler equations are discussed by Tinsley (1997). One is to replace the single-parameter, exponential distribution of stochastic delays in Calvo (1983) with distributions of stochastic responses that allow additional parameters (a nested case is the negative binomial distribution). A second interpretation is that agents may aim to smooth weighted moving averages of decision variables. A representative specification of the latter is when costs are associated with both the levels of assets and changes in time differences (discrete-time derivatives) of assets. Examples of estimated decision rules with more than two eigenvalues include many studies of inventory behavior, including Blanchard (1983), Callen, Hall, and Henry (1990), and Cuthbertson and Gasparro (1993). In this instance, convex penalties are associated with changes in the rate of inventory investment due to production smoothing, given that planned inventory investment is the difference between planned production and expected sales. A more direct example is smoothing moving averages of decision variables when the periodicities of decision making are of lower frequencies than the periodicity of observations, as with seasonal or term contracts.

In many applications, direct observations on average response times of agents can be helpful in assessing the consequences of a polynomial extension of adjustment costs and often support higher-order specifications. In addition, significance tests of friction parameters associated with lags of the dependent variable in error correction formulations of the optimal decision rule are useful in testing the standard assumption that adjustment costs apply only to changes in the level of the decision variable. Use of both sources of information is illustrated in a later section

The next section indicates that polynomial characterizations of frictions can be extended to multiple decision variables to derive a general class of vector rational error correction models.

3. Multiple Decision Variables with Polynomial Adjustment Costs

This section develops error correction formulations of intertemporal decision rules where the prior that convex frictions apply only to levels of decision variables is replaced by a more general specification of frictions. The analysis is also extended to the case of multiple decision variables.

3.1 Derivation of the Euler equations

Denote the $p \times 1$ vector of decision variables by y_t, and the corresponding vector of equilibrium targets by y_t^*. The criterion that agents seek to minimize is extended to:

$$\mathbf{C_t} = E_{t-1}\{\sum_{i=0}^{\infty} B^i[(y_{t+i} - y_{t+i}^*)' C_0 (y_{t+i} - y_{t+i}^*) + (C(L)y_t)'(C(L)y_t)]\}, \tag{8}$$

where the cost function now contains a quadratic function of a matrix lag polynomial of the decision variables, $C(L)$. As in the standard criterion shown in equation (4) for a single decision variable, we assume $C(1) = 0$ to maintain the distinction between targets and frictions. The system of Euler equations for the criterion is

$$E_{t-1}\{(C_0 + C_0')(y_{t+i} - y_{t+i}^*) + C(L)'C(BF)y_{t+i}\} \;\; = \;\; 0.$$

The reciprocal structure of this system of first-order conditions is illustrated for two representative criteria. In the case of smoothing restrictions on moving averages of the decision variables, the adjustment cost portion of the criterion is $(C(L)y_t)'(C(L)y_t) \equiv \sum_{k=1}^{m}((1-L^k)y_{t+i})'C_k((1-L^k)y_{t+i})$ with the associated Euler equation system

$$E_{t-1}\{(C_0 + C_0')(y_{t+i} - y_{t+i}^*) + \sum_{k=1}^{m}(C_k + C_k')((1-L^k)(1-B^kF^k)y_{t+i})\} \;\; = \;\; 0.$$

Similarly, in the case of smoothing restrictions on differences (discrete-time derivatives) of the decision variables, the adjustment cost portion of the criterion is $(C(L)y_t)'(C(L)y_t) \equiv \sum_{k=1}^{m}((1-L)^k y_{t+i})'C_k((1-L)^k y_{t+i})$ with the associated Euler equation system

$$E_{t-1}\{(C_0 + C_0')(y_{t+i} - y_{t+i}^*) + \sum_{k=1}^{m}(C_k + C_k')([(1-L)(1-BF)]^k y_{t+i})\} \;\; = \;\; 0.$$

As shown in these examples, the Euler equations are symmetric in L and BF, the roots occur in reciprocal pairs about the discount factor, and the Euler equation system has the alternative representation,[4]

$$E_{t-1}\{A(BF)A(L)y_{t+i} - A(B)A(1)y^*_{t+i}\} \;=\; 0. \tag{9}$$

Note that this format is exactly the same as that used for the two-root Euler equation in (5) but $A(.)$ is now an m-order matrix polynomial with $p \times p$ matrix coefficients in place of the earlier first-order scalar polynomial $\lambda(.)$.

The system of p interrelated decision rules that satisfy the first order condition in (9) is:

$$\begin{aligned} E_{t-1}\Delta y_t \;&=\; -A(1)(y_{t-1} - y^*_{t-1}) + A^*(L)\Delta y_{t-1} \\ &\quad + \Upsilon'_m[I_{mp} - G]^{-1}\sum_{j=0}^{\infty} G^j \Upsilon_m A(B)A(1)E_{t-1}\Delta y^*_{t+j}, \end{aligned} \tag{10}$$

where G is the $mp \times mp$ bottom-row companion matrix associated with $A(BF)$.

$$G = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ -A_m B^m & -A_{m-1}B^{m-1} & -A_{m-1}B^{m-2} & \cdots & -A_1 B \end{bmatrix}.$$

The derivation of equation (10), including the definition of the selector matrix, Υ_m, is provided in section A.1 of the appendix. Aside from computational simplicity, a considerable advantage of this representation for statistical inference is that it delivers a closed form solution that preserves the parameters of the adjustment cost matrix polynomial without transforming to eigenvalue forms or resorting to numerical solution methods.[5]

3.2 Adding a lag companion forecast system for the forcing terms

Two sets of variables are in the agents' information set: the decision variables, y_t and a remaining set of non-decision information variables that reflect the economic environment of the agents, x_t. The information variables may be exogenous or may be subject to feedback influences from the agents' decision variables.

[4]See discussion of self-reciprocal polynomials in Tinsley (1993).

[5]Representative methods for obtaining linear decision rules from Euler equation systems range from partial fractions expansions of the characteristic roots in Hansen and Sargent (1980) to Schur decompositions in Anderson and Moore (1985) and Anderson, Hansen, McGratten, and Sargent (1996).

The dynamic evolution of information variables, x_t, is represented by a reduced form model in which forecasts of x_t depend on the information available to agents at the end of period $t-1$.

$$E_{t-1}x_t = H_x z_{t-1}. \tag{11}$$

The $n \times 1$ information vector, z_{t-1}, contains lagged observations on y_{t-j} and x_{t-j}, $j = 1, \ldots, q$.

In the case of equilibrium path variables, each element of the $p \times 1$ vector, y_t^*, is assumed to be a linear combination of decision and information variables in the agent information set. These p relationships are represented by

$$y_t^* = \Gamma_*' z_t, \tag{12}$$

where the i^{th} column of Γ_* contains the coefficients used to define y_{it}^*. Because equilibrium targets are often defined as functions only of contemporaneous observations, leaving departures from equilibrium to be attributed to adjustment costs, Γ_* is usually quite sparse, with zero coefficients on all lagged information in z_t. In cases where the targets are linearly related to contemporaneous values of exogenous I(1) variables in x_t, it is often convenient to assume that the equilibrium paths are identified by prior cointegration analysis and included explicitly in the information vector, z_t; in these instances, Γ_* is simply a selector matrix, much like Υ_m, containing relevant unit and zero elements.

Finally, to formally account for motion of the decision variables, y_t, the decision rules in the system (10) are denoted by

$$E_{t-1}y_t = H_y z_{t-1}. \tag{13}$$

Of course, the matrix, H_y, is tightly restricted, as shown below.

The two reduced form forecast models in (11) and (13) can be stacked to define the companion form of the full forecast model

$$E_{t-1}z_{t+i} = H^{i+1} z_{t-1}. \tag{14}$$

Substituting the equations for equilibrium targets from (12) and the forecasts of variables in the information set from (14) into the decision rule system, (10), shows the relationship between planned movements in the decision variables and the agent information set

$$\begin{aligned} E_{t-1}\Delta y_t = & -A(1)(y_{t-1} - y_{t-1}^*) + A^*(L)\Delta y_{t-1} \\ & +[\Upsilon_m'(I_{mp} - G)^{-1} \sum_{i=0}^{\infty} G^i \Upsilon_m A(B) A(1) \Gamma_*' H^i (H - I_n)] z_{t-1}. \end{aligned} \tag{15}$$

It is useful to compare the structure of this solution for the set of decision rules to the standard format of an unrestricted vector error correction model,

$$E_{t-1}\Delta y_t \;\; = \;\; -A(1)(y_{t-1} - y^*_{t-1}) + A^*(L)\Delta y_{t-1} + H^*_{yz} z_{t-1}. \tag{16}$$

In conventional error corrections, H^*_{yz} is an unknown $p \times n$ matrix whose nonzero elements are coefficients of series in the agent information set, z_{t-1}, including lagged targets and information variables. As with the error correction equation (3) in section 2, contributions of lagged regressors in the information vector, z_{i-1}, are often interpreted as decision variable responses to forecasts of the economic environment of agents although no explicit use is made of the forecast model, equation (14).[6] Note that if equilibrium forcing variables are not independent of feedback effects from decision variables, then H^*_{yz} may also contain nonzero coefficients of lagged decision variables. Within the standard vector error correction format, it is impossible to disentangle the contributions of adjustment costs, summarized in the terms associated with the matrix polynomial, $A(.)$, and lagged feedback effects on the forcing terms.

Inspection of the last term in (15) indicates that the lengthy expression in the brackets, [.], likewise defines the coefficients of the information vector, z_{t-1}, for rational decision rules. Thus, the set of decision rules can be rewritten in a compact format similar to that in equation (16).

$$E_{t-1}\Delta y_t \;\; = \;\; -A(1)(y_{t-1} - y^*_{t-1}) + A^*(L)\Delta y_{t-1} + H_{yz} z_{t-1}, \tag{17}$$

where each row of H_{yz} prescribes the influence of variables in the information set on a particular decision variable. However, in contrast to matrix H^*_{yz} in the unrestricted vector error corrections of (16), each row of the matrix H_{yz} in the decision rule equations of (17) is tightly restricted.

To show the restrictions imposed on H_{yz}, the j^{th} row is extracted by premultiplying H_{yz} by a $1 \times p$ selector vector, ι'_j,[7] and simplifying, using column stacks.

$$\begin{aligned} h_j{}' &= vec[(\iota'_j H_{yz})'], \\ &= vec[(H' - I_n)\sum_{i=0}^{\infty}(H')^i \Gamma_* A'(1) A'(B) \Upsilon'_m (G')^i (I_{mp} - G')^{-1} \Upsilon_m \iota_j], \end{aligned}$$

[6]To reduce notation, we ignore restrictions in H^*_{yz} associated with first-differences of regressors, such as the coefficient sum restriction matrix, R_b, shown earlier in section 2. First-difference transformations are captured in equation (15) by the $(H - I_n)$ term.

[7]The selector vector, ι_j has a one in the j^{th} element and zeroes elsewhere.

$$= \quad [\iota_j' \Upsilon_m' (I_{mp} - G)^{-1} \otimes (H' - I_n)][I_{mnp} - G \otimes H']^{-1}[I_{mp} \otimes \Gamma_* A'(1) A'(B)] vec[\Upsilon_m']. \quad (18)$$

This closed form expression for the elements of the $n \times n$ coefficient matrix, H_{yz}, of the information vector is useful in several contexts: First, it provides an explicit expression for the effects of the overidentifying rational expectations restrictions on the forecast role of variables in the information set. Second, it indicates that all of the elements of H_{yz} are known if the elements of the companion matrix of the forecast system, H, and the elements of the companion matrix of the adjustment cost polynomial, G, are known. Third, although the adjustment cost and forecast model parameters are nonlinearly embedded in (18), exact analytical gradients can be used in developing estimators of the unknown parameters.

[

Insert Table 1 here

]

Table 1 contrasts the number of unrestricted coefficients in a conventional vector error correction model, equation (16), with the number of free parameters in a vector rational error correction model, equation (17). The top half of the table indicates the conditioning dimensions, such as the number of decision variables, p and number of forcing variables, $n_v - p$. As shown in the first line of the lower portion of Table 1, the number of unrestricted coefficients in the conventional vector error corrections (VEC) is $pn_vq \equiv p^2q + p(n_v - p)q$, where q denotes the number of lags in the VEC model. By contrast, as indicated in the next line of Table 1, the number of free parameters in the decision rule equations (VREC) is substantially smaller, p^2m. There are two sources of prior restrictions in the rational decision rules (VREC) that reduce the number of free parameters. One is that the maximum degree of the matrix friction polynomial, $A(\,.\,)$, is typically much smaller than the maximum lag of the error correction equations, $m << q$. The other is that responses of decision variables to forecasts of information variables, captured by the coefficient matrix, H_{yz}, of the decision rules, are fully defined by the friction parameters and the forecast model, whereas the corresponding coefficient matrix, H^*_{yz}, of the conventional error corrections is unknown.

4. Estimation of Vector Rational Error Correction (VREC) Systems

As indicated in the preceding section, the forecast model of forcing terms often can be reasonably approximated by a system that is linear in the unknown parameters. However, the requirement for forward forecasts imposed by convex frictions and the assumption of rational expectations lead to decision rules whose parameters are complicated nonlinear functions of both the parameters in the forecast model, H, and the parameters in the frictions polynomial, $A(.)$. Thus, there are often significant computational advantages in separating estimation of the forecast model from estimation of the decision rules.

4.1 Estimating the forecast model

Turning first to the forecast model for non-decision variables, the model for current information variables in (11) can be restated as

$$E_{t-1}x_t = (z'_{t-1} \otimes I_{n_x})h_x, \tag{19}$$

where $h_x \equiv vec(H_x)$ and n_x is the number of current values, which is presumably much smaller than the full set of lagged variables in the $n \times 1$ information vector, z_{t-1}. Denoting the marginal likelihood of the parameters of the forecast model by $L_x(h_x)$, estimates of the parameters are obtained by solving the system of likelihood gradients, $g_x(\hat{h}_x) = 0$. In all cases considered here, these equations are linear in the unknown parameters, h_x, and can be estimated by GLS.

4.2 Issues in estimating the decision rules

The second stage of estimation is aimed at the unknown parameters in the optimal response rules for the decision variables, y_t. The restricted decision rules in (13) are represented in the form

$$E_{t-1}y_t = (z'_{t-1} \otimes I_p)h_y, \tag{20}$$

where h_y is the column stack of the $p \times n$ matrix, H_y. Three issues are discussed regarding the estimation of decision rules.

First, a nontrivial computational issue for models of moderate dimensions is that the elements of H_y are nonlinear functions of the forecast model and the adjustment cost parameters. We indicate this dependence by the notation, $H_y(\theta, h_x)$, where θ is now used to denote the $mp^2 \times 1$ vector of the unknown adjustment cost parameters in the matrix polynomial, $A(.)$ and h_x is the column stack of the forecast model parameters, as referenced in equation (19). The second stage of the estimation proceeds by forming the marginal likelihood of the adjustment cost parameters, $L_y(\theta)$. Then, differentiating with respect to θ, the nonlinear likelihood gradients, $g_y(\hat{\theta}|\hat{h}_x)$, are solved for estimates of the adjustment cost parameters. These gradients are conditioned on first-stage estimates of the forecast model parameters, $\hat{h}_x$, and are presented in section A.2 of the appendix.

A second issue is the introduction of feedback effects of decision variables on forcing terms of the Euler equations. In this case, forecasts of forcing terms depend on future paths of decision variables. This self-referencing characteristic is noted by rewriting the elements of the decision rules as $H_y(\theta, H(h_x, h_y))$. This notation indicates that the matrix of decision rule parameters, H_y, is a nonlinear function of itself, where the last argument is the column stack of decision rule parameters, $h_y \equiv vec(H_y)$. VREC system estimation when the forcing terms include feedback effects of decision variables is discussed in section A2.2 of the appendix.

The final issue is the well-known problem that two-step estimation procedures can understate sampling errors. Specifically, the sampling errors of the adjustment cost parameters need to be adjusted to account for the sampling uncertainty of the first-step estimates of the forecast model parameters that are used to initialize the second-step gradients. Murphy and Topel (1985) cite empirical examples where uncorrected t-ratios of two-step maximum likelihood estimates are overstated by more than 100%. The next subsection, adapted from Tinsley (1997), discusses the issue of correcting for any "generated regressor" bias in the estimated sampling errors of $\hat{\theta}$ when these second-step estimates are conditioned on the first-step estimates of the forecast model, $\hat{h}_x$.

4.3 Accounting for the sampling variability of the forcing term forecast model

To introduce a more compact notation, let β denote the concatenation of the unknown parameters of the forecast model, h_x, and the adjustment cost polynomial, θ. Similarly, $g(\hat{\beta})$ will denote the stack of the relevant likelihood gradients, $g_x(\hat{h}_x)$ and $g_y(\hat{\theta})$.

For a T-period sample, the mean-value theorem implies that the $\sqrt{T}$-normalization of the difference between the sample estimate, $\hat{\beta}$ and the plim, β^o, is provided by rearranging the first-order expansion of the gradients around $g(\beta^o)$,

$$\sqrt{T}(\hat{\beta} - \beta^o) = [-\frac{1}{T}\nabla g(\beta^*)]^{-1}[\frac{1}{\sqrt{T}}g(\beta^o)],$$

where ∇ denotes the gradient of g with respect to β', and the rows of $\nabla g(\beta^*)$ are evaluated at β^*, on the segment connecting $\hat{\beta}$ and β^o. If the normalized Hessian approaches a fixed limit, $plim[-\frac{1}{T}\nabla g(\beta^*)] \rightarrow M$, and the marginal likelihood functions satisfies standard regularity conditions then, as demonstrated in White (1994), the likelihood estimates are distributed asymptotically as the normal distribution,

$$\sqrt{T}(\hat{\beta} - \beta^o) \overset{a}{\sim} N(0, M^{-1}V(M')^{-1}), \tag{21}$$

where V denotes the expected value of the gradient covariance, $V = E\{[\frac{1}{\sqrt{T}}g(\beta^o)][\frac{1}{\sqrt{T}}g(\beta^o)]'\}$.

The advantage of the two-step estimation approach is that the structure of the sampling errors in (21) can be substantially simplified regardless of the feedback characteristics of the decision variables. To see this, partition M and V to reflect the separate contributions of the forecast coefficients, h_x, and the adjustment cost parameters, θ.

$$M = \begin{bmatrix} M_{h_x,h_x'} & M_{h_x,\theta'} \\ M_{\theta,h_x'} & M_{\theta,\theta'} \end{bmatrix}, \qquad V = \begin{bmatrix} V_{h_x,h_x'} & V_{h_x,\theta'} \\ V_{\theta,h_x'} & V_{\theta,\theta'} \end{bmatrix}.$$

Even when forecast equations for information variables include feedback effects of lagged decision variables, the equations for x_t are not functions of the adjustment cost parameters, θ. Thus, the upper right hand partition of M is zero, $M_{h_x,\theta'} = 0$. This, in turn, implies that the inverse of M required for equation (21) simplifies to

$$M^{-1} = \begin{bmatrix} M^{-1}_{h_x,h'_x} & 0 \\ -M^{-1}_{\theta,\theta'} M_{\theta,h'_x} M^{-1}_{\theta,\theta'} & M^{-1}_{\theta,\theta'} \end{bmatrix}.$$

Substituting this partitioned inverse into the asymptotic covariance indicated in (21) yields the following expression for the asymptotic covariance of the adjustment cost parameters,

$$\begin{aligned} var(\hat{\theta} - \theta^{\circ}) \quad = \quad & M^{-1}_{\theta,\theta'} + M^{-1}_{\theta,\theta'}[M_{\theta,h'_x} M^{-1}_{h_x,h'_x} M'_{\theta,h'_x} \\ & -V'_{h_x,\theta'} M^{-1}_{h_x,h'_x} M'_{\theta,h'_x} - M_{\theta,h'_x} M^{-1}_{h_x,h'_x} V_{h_x,\theta'}] M^{-1}_{\theta,\theta'}. \end{aligned}$$

The first term, $M^{-1}_{\theta,\theta'}$, is the covariance of the adjustment cost parameters provided by the two-step estimation procedure, without correction for "generated regressor" bias. The remaining terms define the adjustment to account for the sampling variability of the estimated parameters of the forecast model, $\hat{h}_x$, where all moments are evaluated by sample estimates such as the GLS estimate of the covariance of the forecast model parameters, $M^{-1}_{h_x,h'_x}$.

5. An Empirical Application: Equipment Investment

This section will use the VREC methodology to empirically examine a simple model of aggregate investment under uncertainty. The model adopts a time-to-build technology assumption in the spirit of Kydland and Prescott (1982) but also assumes convex costs of adjusting the rate of installed equipment.

The target for investment satisfies long-run profit maximization in the absence of adjustment costs. For a Cobb-Douglas production function, the first-order condition is, in logarithmic form,[8]

$$log(\alpha_K) + y - k + log(P^Y / rP^K) \quad = \quad 0,$$

where α_K is the steady-state capital share, P^Y is the output price, y is log output, k is the log capital stock, r is the return to capital, and P^K is the price of capital. Based on this

[8] In this section, i_t and k_t will denote the logs of capital investment, I_t and the capital stock, K_t, respectively.

condition, the log of target capital, k_t^*, is

$$k_t^* \;=\; log(\alpha_K) + y_t - log((r_t P_t^K)/P_t^Y). \tag{22}$$

Under time-to-build constraints, a fraction ψ_j of investment expenditure in each period is allocated to capital that is j periods from completion, $j = 1, \ldots, m$ and $\sum_{j=1}^{m} \psi_j = 1$. The investment expenditure in t on capital that is j periods from completion is denoted by $\psi_j I_t$, where I_t is total investment expenditure in t. Thus, installations of capital equipment in t are denoted by $\chi_t = \sum_{j=1}^{m} \psi_j I_{t-j}$, which determines the accumulation rate of productive capital, $K_t = (1-\delta)K_{t-1} + \chi_t$.

In the steady state, the log of the installation rate of productive capital equipment, $\log(\chi)$, equals log investment, i, and satisfies

$$i \;=\; log(g+\delta) + k,$$

where g is the average growth rate of output and δ is the average depreciation rate of the capital stock. The evolution of the log investment target, i_t^*, based on this steady state condition is

$$i_t^* = \frac{1+g}{g+\delta} g_t + \frac{1-\delta}{g+\delta}\delta_t + k_t^*, \tag{23}$$

where $g_t \equiv \Delta y_t$ denotes output growth and the first two terms on the right hand side of equation (23) approximate $\log(g_t + \delta_t)$.

Finally, convex adjustment costs are assumed for changes in the installation rate of capital

$$c(\Delta\chi_t)^2 \;\equiv\; c_1(\psi(L)\Delta i_t)^2,$$

using the approximation, $(e^{i_t} - e^{i_{t-1}}) \approx \Delta i_t$.

Thus, producers choose the rate of investment that solves the stochastic optimization:

$$i_t \;=\; argmin\left(E_{t-1}\{\sum_{j=0}^{\infty} B^j[(i_{t+j} - i_{t+j}^*)^2 + c_1(\sum_{s=1}^{m}\psi_s \Delta i_{t+j-s})^2]\}\right), \tag{24}$$

where polynomial adjustment costs are a consequence of the embedded time-to-build constraints within the convex costs of adjusting the installation rate of productive capital. The criterion in (24) also encompasses a model of capital investment without polynomial frictions. That is, the prior that convex frictions apply only to changes in the current level of the decision variable is obtained by setting m = 1. The empirical consequences of relaxing this conventional prior are explored by estimating decision rules that differ only in the order of the friction polynomial, m.

The producer decision rule for capital equipment is estimated for the sample span 1971q1-1997q3. Quarterly data from the U.S. National Income and Product Accounts are used for producers' durable equipment, i_t, business output, y_t, and the depreciation rate of durable equipment, δ_t, The relative rental rate of capital, $(r_t P_t^K)/P_t^Y$, is taken from the data bank of the FRB/US model of the U.S. economy, Brayton and Tinsley (1996). The quarterly discount factor, B, is set to .98, consistent with the postwar return to U.S. equity of about 8 percent.[9] The investment target series, i_t^*, is constructed using the definition in equation (23) where required coefficients are measured by sample moments.[10] An ADF test of the cointegrating discrepancy, $i_t - i_t^*$, rejects the null hypothesis of a unit root at a significance level that exceeds 99%. The arguments of the q-order VAR forecast model are business output, y_t, the equipment depreciation rate, δ_t, and the rental rate of capital equipment, $(r_t P_t^K)/P_t^Y$, with $q = 8$.

Insert Table 2 here

Results of estimating the decision rule for producers' durable equipment are presented in Table 2. All t-ratios, shown in parentheses, are adjusted for sampling errors associated with finite-sample estimates of the VAR forecast model, as discussed in the preceding section. The first equation listed in Table 2 illustrates the conventional assumption that adjustment costs apply only to changes in the level of the decision instrument, $m = 1$. The error correction coefficient, $A(1) = -.073$, is significantly different from zero. The associated mean lag response of investment to unanticipated shocks is 12.6 quarters or about three and one-half years. This seems lengthy for equipment responses given that the average useful lifetime of equipment is only about seven years. By contrast, a study of the capital investments of individual firms by Schaller (1990) indicates a total response lag of about one year. The mean lag from manufacturing new orders to shipments of machinery and equipment is about two quarters, using a monthly sample from 1971-1997. However, this source of external lags

[9] Results are insensitive to moderate variations in the discount factor.

[10] The sample means, $\hat{g}$ and $\hat{\delta}$, imply $(1+\hat{g})/(\hat{g}+\hat{\delta}) = 7.3$ and $(1-\hat{\delta})/(\hat{g}+\hat{\delta}) = 6.3$.

does not account for additional internal delays such as installation lags.

Two additional tests of the estimated two-root decision rule, $m = 1$, also suggest problems in dynamic specifications. The Breusch-Godfrey test statistic, $BG(12)$, indicates the null hypothesis of serially independent residuals is rejected with a p-value of .00. The test of the rational expectations overidentifying restrictions on the coefficient vector of the agent VAR forecast model indicates these restrictions are rejected also with a p-value of .00. As noted in Tinsley (1993) and Ericsson and Irons (1995), serially correlated residuals and rejections of RE restrictions are not atypical for applications of RE decision rules to macroeconomic aggregates.

The second equation in Table 2 lists properties of an estimated investment decision rule that is subject to a higher-order polynomial description of frictions. The coefficients of two additional lags of investment are statistically different from zero.[11] This is a six-root decision rule, $m = 3$, using three eigenvalues to discount forward expectations of planned investment, $E_{t-1}\{i^*_{t+j}\}$.

The dynamic properties of the second estimated decision rule are quite different. The mean lag is about four quarters, which is more compatible with other estimates of response times for capital equipment. Neither the null of serially independent residuals (p-value of .63) nor the RE overidentifying restrictions (p-value = .15) are rejected at conventional levels of significance.

Insert Table 3 here

Another way of interpreting the role of the additional characteristic roots in the higher-order decision rule is presented in Table 3. The decision rule for capital equipment can be reformulated as

$$i_t \quad = \quad E_{t-1}\{\sum_{j=-\infty}^{\infty} w(j) i^*_{t+j}\}. \tag{25}$$

to indicate the lag responses of current investment to unexpected shocks $(w(j), j < 0)$ and lead responses to expected future investment plans, $(w(j), j \geq 0)$. The first line of Table 3 lists lag and lead response weights for the standard two-root decision rule ($m = 1$). The

[11] Significance tests of additional lags of the dependent variable do not support $m > 3$.

forward response weights, $(w(j), j \geq 0)$, of the two-root rule are determined by powers of a single root, as are the backward response weights. Thus, the response weights of the conventional two-root decision rule resemble a two-sided geometric distribution with the largest weight, $w(0)$, on the current quarter, t. The weights are slightly asymmetric due to the discounting of future periods over the planning horizon.

The second line of Table 3 lists the lag and lead weights for the higher-order decision rule ($m = 3$). The principal effect of the additional friction parameters is to increase the size of the response weights in a four-quarter neighborhood of the current quarter. Specifically, the weights from $w(-4)$ to $w(4)$ of the 6-root decision rule are about twice the size of the corresponding geometric response weights associated with the two-root rule. Because the sum of the response weights is unity, the larger response weights near the current quarter, t, imply much smaller responses than the geometric weights of the two-root decision rule for the remaining periods that are more distant from the current quarter. Generally, as in Tinsley (1997), relative larger responses to nearby events appear to be typical of polynomial frictions applied to macroeconomic decision variables.

6. Concluding Remarks

This paper proposes vector rational error correction models as a useful intersection of atheoretic time series models and rational agent models. Vector rational error correction models inherit the data fitting objectives of VAR models but significantly reduce the number of free parameters by incorporating rational agent restrictions. Likewise, rational error corrections provide linear-quadratic approximations of rational agent theories but facilitate statistical testing of these theoretical priors.

The suggested extension of adjustment cost smoothing to a polynomial in the lag operator means that lags in the first differences of decision variables appear as regressors, in addition to the contributions of lagged decision variables in error correction terms. As illustrated in section 5, empirical applications generally suggest that this polynomial generalization is very important in obtaining a better match of data moments and a markedly lower rejection rate of rational expectations restrictions.[12]

A matrix polynomial formulation of smoothing penalties may be useful in other applications, Kozicki (1996). An example is a multivariate construction of unobserved trend

[12] See descriptions of additional empirical examples of decision rules with polynomial frictions applied to a wide range of macroeconomic aggregates in Brayton and Tinsley (1996) and Tinsley (1997).

and cycle components. In a vector analogue of Hodrick and Prescott (1997), suppose a time series vector, y_t, is viewed as the sum of a vector of smoothly-varying trend components, $K\tau_t$, and a vector of trend deviations. The matrix, K, need not be square in the case of common trends. The trend components can be interpreted as "decision variables" that minimize a multivariate analogue of the Hodrick-Prescott criterion.

$$\tau_t \quad = \quad argmin[\sum_{t=1}^{T}(y_t - K\tau_t)'C_0(y_t - K\tau_t) + \lambda\sum_{k=1}^{m}(K(1-L^k)\tau_t)'C_k(K(1-L^k)\tau_t)].$$

As with the case of restricted movements in decision variables, the trend components will be less volatile than the original time series if changes in moving averages (or higher-order differences) of the trend components are penalized.

For simplicity, the exposition of this paper assumes the target paths of decision variables are described by cointegrations. It is straightforward to accommodate theories that specify stationary, I(0), arguments of the conditional equilibrium paths; see examples in Tinsley (1993). This variation introduces discounted paths of expected future values of the stationary variables into decision rules.

A distinguishing feature of the vector formulation of rational error correction decision rules is the introduction of lagged error correction "gaps" from other decision variables. Although empirical estimates of interrelated factor demand models are seldom reported in macroeconomic literature, it seems plausible that agents with multiple decision variables will choose systematic multivariate feedback responses to equilibrium displacements in their own sector and in other sectors. Empirical work on persistent cross-sectoral effects reported in Konishi and Granger (1992) and Konishi, Granger, and Ramey (1992) suggests that vector rational error corrections may provide a framework for theoretical interpretations of these empirical interactions.

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Appendix Derivation and Estimation of the VREC System

A.1. Derivation of the vector error correction system of decision rules, equation (10)

This section of the appendix derives the system of interrelated decision rules that satisfy the first-order conditions in (9). The derivation follows that in Tinsley (1993) which uses two companion systems. One is a lead companion system associated with the lead matrix polynomial, $A(BF)$, and the other is a lag companion system formulation of the agent forecast model of the forcing terms, y^*.

To obtain the forward-looking format characteristic of intertemporal decision rules, multiply through by the inverse of the factor polynomial in the lead operator, $A(BF)^{-1}$.

$$E_{t-1}A(L)y_t \;=\; E_{t-1}A(BF)^{-1}A(B)A(1)y_t^*. \tag{26}$$

The forcing term of this equation

$$v_t^e \;\equiv\; E_{t-1}A(BF)^{-1}A(B)A(1)y_t^*,$$

is generated by the lead vector autoregression system

$$A(BF)v_t^e \;=\; E_{t-1}A(B)A(1)y_t^*. \tag{27}$$

To recast this lead autoregressive model into a more convenient first-order format, define two $mp\times 1$ column vectors: $w_t \equiv [v^e_{t+m-1}{}', \cdots, v_t^{e\prime}]'$ and $f_t \equiv [0_p', \cdots, 0_p', (A(B)A(1)E_{t-1}y_t^*)']'$. Also, as an aid to later discussion, note that the second column vector can also be represented by $f_t \equiv \Upsilon_m A(B)A(1)E_{t-1}y_t^*$, where Υ_m is an $mp\times p$ selector matrix with a pth-order identity matrix in the bottom p rows and zeroes elsewhere.[13] Using these vectors, the companion form of the lead autoregression in (27) can be compactly represented by

$$\begin{aligned} w_t &= Gw_{t+1} + f_t, \\ &= \sum_{j=0}^{\infty} G^j f_{t+j}, \end{aligned} \tag{28}$$

where the $mp \times mp$ companion matrix, G, is defined in section 3.

The solution for the current forcing term can be recovered by using the selector matrix, $v_t^e = \Upsilon_m' w_t$. Substituting this solution into (26) yields the system of p decision rules shown

[13]That is, $\Upsilon_m \equiv (\iota_m \otimes I_p)$ where, in turn, ι_m is an $m \times 1$ selector vector with a one in the bottom element and zeroes elsewhere.

earlier in (10):

$$\begin{aligned}
E_{t-1}\Delta y_t &= -A(1)y_{t-1} + A^*(L)\Delta y_{t-1} + \Upsilon'_m \sum_{j=0}^{\infty} G^j \Upsilon_m A(B)A(1)y^{e*}_{t+j}, \\
&= -A(1)y_{t-1} + A^*(L)\Delta y_{t-1} \\
&\quad + \Upsilon'_m [I_{mp} - G]^{-1}[\Upsilon_m A(B)A(1)y^*_{t-1} + \sum_{j=0}^{\infty} G^j \Upsilon_m A(B)A(1)E_{t-1}\Delta y^*_{t+j}], \\
&= -A(1)(y_{t-1} - y^*_{t-1}) + A^*(L)\Delta y_{t-1} \\
&\quad + \Upsilon'_m [I_{mp} - G]^{-1} \sum_{j=0}^{\infty} G^j \Upsilon_m A(B)A(1)E_{t-1}\Delta y^*_{t+j}, \qquad (29)
\end{aligned}$$

In the first line of (29), the lag matrix polynomial, $A(L)$, is partitioned into a level and difference format, $A(L) = A(1)L + (I - A^*(L)L)(1-L)$ where $A^*(.)$, is an (m-1)-order matrix polynomial.[14] The second line in (29) partitions the forward paths of the forcing terms into an initial level and forward differences. The third line uses the identity, $\Upsilon'_m[I-G]^{-1}\Upsilon_m \equiv A(B)^{-1}$, to isolate the error correction "gaps," $y_{t-1} - y^*_{t-1}$. The final term in the last line captures the present-value effects of expected forward differences in the forcing terms. In contrast to the single discount factor of forward expectations in the standard two-root decision rule discussed in section 2, there are now mp discount factors associated with the eigenvalues of the $mp \times mp$ companion matrix, G.

A.2. VREC system estimation

A.2.1 Forcing terms are independent of decision variables

In the first stage of estimation, we assume the forecast model of the non-decision variables is linear in the unknown parameters. Assuming normally distributed forecast errors, the log of the marginal likelihood of the forecast model parameters is

$$L_x(h_x) = -(T/2)log(|V_{\epsilon x}|) - (Tn_x/2)log(2\pi) - \sum_{t=1}^{T}(x_t - H_x z_{x,t-1})' V_{\epsilon x}^{-1}(x_t - H_x z_{x,t-1}), \qquad (30)$$

where $V_{\epsilon x} \equiv E[(x_t - H_x z_{x,t-1})(x_t - H_x z_{x,t-1})']$, and the subscript convention indicates that the information vector for the forecast model of the forcing terms, $z_{x,t-1}$, now excludes lagged decision variables. Using notation introduced in equation (19), first-stage estimates of the

[14]Component matrices in $A^*(.)$ are sums of the matrices in $A(L)$.

coefficients of the forecast equations for x_t are the solutions of the following gradients of equation (30):

$$\begin{aligned} g_x(\hat{h}_x) &= \sum_{t=1}^{T}(z'_{x,t-1} \otimes I_{n_1})V_{\epsilon x}^{-1}(x_t - \hat{H}_x z_{x,t-1}) \\ &= 0. \end{aligned}$$

These are equivalent to GLS estimates of the forecast model equations, or even OLS estimates if the equations have identical regressors.

Turning to the second-stage estimation of the adjustment cost parameters, and again assuming normally distributed errors, the log of the marginal likelihood of θ is

$$L_y(\theta) = -(T/2)log(|V_{\epsilon y}|) - (Tp/2)log(2\pi) - \sum_{t=1}^{T}(y_t - H_y z_{t-1})'V_{\epsilon y}^{-1}(y_t - H_y z_{t-1}), \quad (31)$$

where $V_{\epsilon y} \equiv E[(y_t - H_y z_{t-1})(y_t - H_y z_{t-1})']$ and the information vector now includes lagged values of the decision variables to account for the adjustment cost terms associated with the matrix polynomial, $A(.)$. Maximum likelihood estimates of the adjustment cost parameters, $\hat{\theta}$, are defined by the zeroes of the gradients of (31) with respect to θ, given the first-stage estimates of the forecast model coefficients, $\hat{h}_x$.

$$\begin{aligned} g_y(\hat{\theta}) &= \sum_{t=1}^{T}[\partial vec(H_y(.|\hat{h}_x))/\partial\theta']'(z'_{t-1} \otimes I_p)V_{\epsilon y}^{-1}(y_t - H_y(\hat{\theta}|\hat{h}_x)z_{t-1}) \\ &= 0. \end{aligned} \quad (32)$$

The brackets, [.], contain the derivatives of the decision rules with respect to the adjustment cost parameters of the matrix polynomial, $A(.)$, and its associated companion matrix, G. The format of these nonlinear gradient equations is well-suited for estimation by Gauss-Newton iterations. Details of this approach illustrating the use of exact analytical gradients of REC decision rules are provided in Tinsley (1997).

The gradient solutions for the adjustment cost parameters are conditioned on the first-step estimates of the parameters of the forecast model for the forcing terms, $\hat{h}_x$. Thus, as discussed earlier, the sampling errors of the estimated adjustment cost parameters, $\hat{\theta}$, must be adjusted for the uncertainty of the first-step estimates. This requires constructing the partitions of the Hessian, M, in (21). As indicated below, with the one exception of additional gradients of the likelihood of adjustment cost parameters with respect to the vector of forecast model parameters, $\partial L_y/\partial h_x$, these partitions can be assembled from gradients and covariances already produced in the first and second stages of estimation.

$$
\begin{aligned}
M_{h_x,h_x'} &= E[-\tfrac{\partial^2 L_x}{\partial h_x \partial h_x'}] &= E[g_x(h_x)g_x(h_x)']. \\
M_{h_x,\theta'} &= E[-\tfrac{\partial^2 L_x}{\partial h_x \partial \theta'}] &= 0. \\
M_{\theta,\theta'} &= E[-\tfrac{\partial^2 L_y}{\partial \theta \partial \theta'}] &= E[g_y(\theta)g_y(\theta)']. \\
M_{\theta,h_x'} &= E[-\tfrac{\partial^2 L_y}{\partial \theta \partial h_x'}] &= E[g_y(\theta)(\tfrac{\partial L_y}{\partial h_x'})'].
\end{aligned}
\tag{33}
$$

A.2.2 Forcing terms include feedback effects of decision variables.
Finally, consider the case of feedback effects from decision variables to the forcing terms of the Euler equations. In brief, this means that forecasts of the forward positions of forcing terms incorporated in agent decision rules now require also accompanying forecasts of the decision variables.

As a consequence, the decision rules, $H_y z_{t-1}$, are now included in additional rows of the forecast model matrix of reduced form coefficients, denoted by H in the earlier derivation of the decision rules. This implies that the decision rules are self-referencing, which is indicated by the notational convention

$$y_t = H_y(\theta, H(h_x, h_y))z_{t-1}, \tag{34}$$

where there are now two components of the forecast model: h_x, the coefficient stack of the forecast model for the non-decision information variables, x_t; and h_y, the coefficient stack of the p rows of the forecast model for the decision variables. Under rational expectations, the vector, h_y, is equal to the coefficient stack of the decision rules.

To address this self-referential mapping, the partial derivatives of the decision rule coefficients with respect to the adjustment cost parameters are represented as

$$
\begin{aligned}
[\partial vec(H_y)/\partial\theta'] &= [\partial vec(H_y(.|\hat{H}))/\partial\theta'] + [\partial vec(H_y(\hat{\theta}, H(\hat{h}_x, h_y)))/\partial h_y'][\partial h_y/\partial\theta'], \\
&= [I - [\partial vec(H_y(\hat{\theta}, H(\hat{h}_x, h_y)))/\partial h_y']]^{-1}[\partial vec(H_y(.|\hat{h}_x, \hat{h}_y))/\partial\theta'].
\end{aligned}
\tag{35}
$$

In the first line, the initial term on the right hand side of the equal sign holds the forecast model fixed when differentiating with respect to the adjustment cost parameters, θ. The second term identifies the effect of differentiation through the decision rule channels in the forecast model by the chain rule of differentiation. Because the matrix $[\partial h_y/\partial\theta']$ is the same as that on the left hand side of the equal sign, the second line provides a closed form resolution of the mapping required by rational expectations.

As a consequence of the feedback effects of decision variables on forcing terms, two alterations are needed in the definition of the decision rule gradients in equation (32). First, rows containing the reduced forms of the decision rules are included in the companion matrix of the forecast model, H. The conditioning of the decision rules on the extended forecast model is indicated by replacing $H_y(\hat{\theta}|\hat{h}_x)$ in equation (32) by $H_y(\hat{\theta}|\hat{H})$. Second, the rational expectations mapping, shown above in equation (35), needs to be incorporated in the term for the partial derivative of the stack of the decision rule coefficients, $[\partial vec(H_y)/\partial\theta']$. Thus, in accounting for feedback effects on forcing terms, the gradient equations for the decision rule parameters in (32) are replaced by:

$$\begin{aligned} g_y(\hat{\theta}) &= \sum_{t=1}^{T}[I - [\partial vec(H_y(\hat{\theta}, H(\hat{h}_x, h_y)))/\partial {h_y}']]^{-1} \times \\ &\quad [\partial vec(H_y(.|\hat{h}_x, \hat{h}_y))/\partial\theta']'(z'_{t-1} \otimes I_p)V_{\epsilon y}^{-1}(y_t - H_y(\hat{\theta}|\hat{H})z_{t-1}) \\ &= 0. \end{aligned} \tag{36}$$

Table 1
Restrictions in Decision Rule Equations[a]

given:		
	total variables	n_v
	decision variables	p
	forcing variables	$n_v - p$
	lags in expectations VAR	q
	order of frictions polynomial	m
implies:		
	parameters in unrestricted error correction equations (VEC)	$p^2q + p(n_v - p)q$
	unrestricted parameters in decision rule equations (VREC)	p^2m
	restrictions in decision rule equations	$p^2(q-m) + p(n_v - p)q$

[a]Parameter count assumes $n_v \geq p$, $q \geq m$. For simplicity, entries do not account for unit root restrictions or cointegration among forcing variables, and excludes constants and coefficients on deterministic trends from free parameter calculations.

Table 2
Decision Rules for Producers' Durable Equipment Investment[a]

$$\Delta i_t = -A(1)[i_{t-1} - i^*_{t-1}] + A^*(L)\Delta i_{t-1} + h'_* z_{x.t-1} + a_t.$$

m^b	$A(1)$	A^*_1	A^*_2	R^2	SEE	mean lag[c]	$BG(12)^d$	$LR(h_*\|z_{t-1})^e$
1	-.073			.24	.028	12.6	.00	.00
	(-3.6)							
3	-.078	.266	.334	.50	.024	4.1	.63	.15
	(-4.6)	(2.8)	(3.4)					

[a]Quarterly data are: log producers' durable equipment investment, i; log target investment, i^*; and the producers' information vector, $z_{x,t-1}$, containing lagged values of the determinants of i^*, including business output, the relative rental rate of producers' durable equipment, and the depreciation rate of durable equipment. The sample span is 1971q1-1997q3. The current investment response to expected growth rates of target investment is denoted by $h'_* z_{t-1}$ (see text).

[b]The number of estimated friction parameters in the $2m$-order decision rule.

[c]In quarters.

[d]Rejection probability of serially independent residual , Breusch-Godfrey test (12 lags).

[e]Rejection probability of RE overidentifying restrictions on the coefficient vector for expected growth rates of target investment, h_*.

Table 3

Selected Lag and Lead Weights for Equipment Investment Decision Rules[a]

$$i_t = E_{t-1}\{\sum_{j=-\infty}^{\infty} w(j) i^*_{t+j}\}.$$

	lag weights				lead weights		
	w(-12)	w(-8)	w(-4)	w(0)	w(4)	w(8)	w(12)
$m = 1$	.017	.023	.031	.042	.029	.020	.013
$m = 3$	.002	.024	.065	.096	.060	.021	.001

[a]Quarterly weights generated by the estimated $2m$-order decision rules of Table 2.